Advances in Portfolio Risk Control.

Risk ! Parity ?

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Abstract

Spurred by the increased interest in applying "risk control" techniques in an asset allocation context, we offer a practitioner's review of techniques that have been newly proposed or revived from academic history. We discuss minimum variance, "1/N" or equal-weighting, maximum diversification, volatility weighting and volatility targeting – and especially "risk parity", a concept that has become a real buzz word. We provide a taxonomy of risk control techniques. We discuss their main characteristics and their pluses and minuses and we compare them against each other and against the maximum Sharpe Ratio criterion. We illustrate their implications by means of an empirical example. We also highlight some important papers from the vast and still growing literature in this field. All in all, this note serves as a practical and critical guide to risk control strategies. It may help you to demystify risk control techniques, to appreciate both the "forest" and the "trees", and to judge these techniques on their potential merits in practical investment applications.

Introduction

Recently there has been increased interest in applying "**risk control**" techniques in an asset allocation context. Some examples of techniques that has been newly proposed or revived from academic history are minimum variance, "1/N" or equal-weighting, maximum diversification, volatility weighting and volatility targeting – and especially "risk parity", a concept that has become a real buzz word.

In this note we provide a **taxonomy** of risk control techniques. We discuss their main characteristics and their **pluses and minuses**, we **compare** them against each other and against the **maximum Sharpe Ratio** criterion – and we illustrate their **implications** by means of a single empirical example that we extend throughout the note. We also highlight some **important papers** from the vast and still growing literature in this field. All in all, this note serves as a practical and critical guide to risk control strategies that may help you to appreciate both the "forest" and the "trees" and to judge these techniques on their actual potential merits in practical investment applications.

The main question in risk control is : **"does it work ?"** Do risk control techniques achieve the *ex ante* targeted risk balance or risk profile ? Can we avoid hot spots (pockets of risk concentration in a portfolio) and can we achieve diversification against losses ? Although these are natural questions to pose in the context of risk control, the current discussions on risk control extend its significance to offering opportunities to reap risk-adjusted **outperformance**. But why would ignoring the return dimension *ex ante* produce portfolios that are superior in terms of *ex post* risk-adjusted performance ?

Several studies indicate that the historical outperformance of risk control strategies can be linked to overweighting asset classes that in the **rear view mirror** have paired high historical risk premia with low risk levels (as is the case for bonds, e.g.) or to implicit exposures to factor premia. However, focusing directly on factor exposures, as is done in **factor investing**, provides a much more efficient and effective way to capture factor premia. Still, focusing only on risk aspects when forming a portfolio is a perfectly sensible starting point when one has only low **confidence in *ex ante* risk premia estimates**. From the perspective of estimation risk, mis-estimation of risk premia has the greatest impact on portfolio composition and especially risk premia are notoriously hard to estimate *ex ante*. For example, suppose that *ex ante* you cannot meaningfully differentiate between all assets' Sharpe Ratios (so you assume that all Sharpe Ratios are equal, implying that all risk premia are proportional to their volatilities), then constructing a Maximum Diversification portfolio gives the maximum Sharpe Ratio portfolio. When, in addition to equal Sharpe Ratios, you cannot meaningfully differentiate between asset correlations (so you also assume that all correlations are uniform), then applying Risk Parity gives the maximum Sharpe Ratio portfolio. So besides the risk dimension, also the potential relevance of risk control techniques in full-fledged risk-return optimization is not to be under-estimated.

Outline

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1. A taxonomy

Risk control strategies serve to control the risk profile of an investment portfolio or investment strategy. Risk is often equated with standard deviation (of total return or differential return with respect to a benchmark), but most results carry over to downside risk measures such as portfolio loss or Value-at-Risk. Apart from being techniques to analyze, monitor and change a portfolio's risk profile, a large part of the literature has promoted risk control as a full-fledged investment criterion -- suggesting that controlling the risk dimension is sufficient to build a portfolio. We revisit this issue when discussing the various risk control strategies in more detail. We start off, however, with sketching a taxonomy of risk control strategies.

The main skeleton of risk control strategies has a time series branch and a cross-section branch.

A. Time series (TS) :

The objective of risk control over time is to control the portfolio risk level **over time**. There are two closely related TS techniques :

- volatility weighting over time : the exposure to (the risky part of) a portfolio is adjusted according to the level of forecasted volatility;
- volatility targeting : this is volatility weighting with the specific goal to achieve a pre-specified level of portfolio volatility.

When weighting or targeting a portfolio's risk level over time, the composition of a portfolio's risky part is not changed -- only the weights of the risky part and the risk free part are adjusted.

B. Cross-section (XS) :

The objective of risk control in the cross-section is to control a portfolio's risk profile **at a given point in time**. The focus is across assets : reweighting the portfolio constituents so as to obtain a desired risk profile. The main XS risk control strategies are :

- 1/N, or the equally-weighted portfolio
- minimum variance portfolio
- maximum diversification portfolio
- Risk Parity, that comes in two flavours :
  - → "Equal Risk Contribution" (ERC) or "full" risk parity
  - → "Inverse Volatility" (IV) or volatility weighting in cross-section.

Finally, we escape from a risk-only perspective and consider the Maximum Sharpe Ratio portfolio.

Before discussing the above techniques in more detail, we outline our empirical example that we'll use to illustrate these techniques and their implications.

2. The empirical example and preliminaries

We consider monthly data over the 10Y period Jun 2002 – May 2012 (120 months) for a selection of US assets classes. See Table 1.

Table 1 : overview of assets and their market cap weight.

Assets :	Abbrev :	Market Cap Index :
Risk free rate of return		
Equities	Eq	45%
Aggregate Treasuries	Tsies	30%
Corporate Investment Grade	IG	20%
Corporate High Yield	HY	5%

Data sources :

- The risk free return comes from the Ibbotson "Stocks, Bills, Bonds and Inflation" database.
- Equities is the market factor from Kenneth French's database.[1]
- The fixed income series are taken from Barclays Live.[2]
- All returns are in USD.

The composition of the market capitalization weighted portfolio "Mkt Cap" is estimated as per 2012Q1.[3] "EqWtd" is the equally-weighted portfolio. The descriptive statistics are given in Table 2 on the next page.

[1] The Ibbotson risk free rate and the market factor can be downloaded from http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[2] Download from https://live.barcap.com/ .

Table 2 : Statistics of US Excess Returns (p.a.) over the risk free rate (Jun 2002 - May 2012).

%	Equities	Tsies	IG	HY	Mkt Cap	EqWtd
Return statistics :						
avge p.a.	4.83	3.93	4.85	7.65	4.71	5.31
stdev p.a.	16.75	4.93	6.42	11.27	8.24	7.30
Sharpe Ratio	0.29	0.80	0.76	0.68	0.57	0.73
Correlations :						
Equities		-0.34	0.30	0.74	0.95	0.87
Tsies			0.52	-0.22	-0.06	0.00
IG				0.59	0.56	0.71
HY					0.79	0.90
Mkt Cap						0.46

Observations :

- Over this historical period, fixed income assets were the real winners. This is not surprising given the substantial tail wind from decreasing interest rates. Especially Tsies paired a substantial average return with a relatively low level of risk.
- Equities showed the highest volatility, but viewing the Sharpe Ratio this was not matched by a proportionally higher risk premium.
- Equities and Tsies were negatively correlated, providing hedge opportunities (see the small negative correlation between Tsies and the market cap portfolio).
- The highest correlation is between Equities and HY, pointing at a high correlation between equity risk and credit risk. Credit risk is dominant in HY and the negative correlation between interest rates and credit spreads manifests itself in the negative correlation between Tsies and HY.

Money allocation versus risk allocation

The **money allocation** in the market cap portfolio is given in Table 1.
For the **risk allocation** within the market cap portfolio, we compute the OLS regression slope or beta of the assets against the market cap portfolio. It can be shown that this beta represents the relative marginal contribution of the corresponding asset to the overall portfolio risk (for details, see the Technical Appendix). The component risk contribution is given by the product of the investment weight and the beta. Hence, the betas can be interpreted as the adjustment factors to transform money allocation into risk allocation (note that the weighted average value of beta is unity). The risk allocation within the market cap portfolio is given in Table 3.

[3] Sources are (1) Securities Industry and Financial Markets Association (SIFMA), US Bond Market Outstanding, download from http://www.sifma.org/research/statistics.aspx, (2)WorldBank, year-end market capitalization of listed companies by country, download from http://data.worldbank.org/indicator/CM.MKT.LCAP.CD and (3) Barclays Live (for the relative IG and HY capitalizations).

Table 3 : Risk attribution with respect to Mkt Cap portfolio.

	Eq	Tsies	USIG	USHY	sum
weight	45%	30%	20%	5%	
beta	1.93	-0.04	0.44	1.08	
% risk contribution	87%	-1%	9%	5%	100%

From Table 3 we see a nasty surprise : the market cap portfolio appears to be a properly diversified portfolio but in reality almost 90% of the risk within that portfolio is due to equities. (This was already forewarned by the high correlation between equities and the market cap portfolio as shown in Table 2.) The same finding is reported for conventional 60/40 equity-bond portfolios in general, and for typical "Yale" portfolios (were commodities and/or alternatives are added to main holdings of equities and bonds).

Although we focus on volatility as the risk measure, the same pattern arises when we consider the average of the 6 largest monthly losses against the risk free rate on the market cap portfolio over this period, see Table 4. Equities also contributed by far the most to the realized losses (where the exact 87% contribution of Equities is a coincidence with Table 3).

Table 4 : Absolute and % contribution of assets to average of 6 largest losses on the market cap portfolio (in terms of excess returns).

Cap Index	Eq	Tsies	USIG	USHY
-5.56	-4.84	0.00	-0.50	-0.21
100%	87%	0%	9%	4%

The extremely large contribution of equities to (downside) risk within portfolios that seem only moderately geared towards equities provided the impetus to the research into risk control strategies. In the remainder of this note, we use this empirical example to illustrate various risk control strategies.

Implied risk premia and the implied Sharpe Ratios

There is one additional perspective we'd like to highlight – a perspective that is helpful in evaluating risk control strategies *vis à vis* the maximum Sharpe Ratio portfolio. For each of the portfolios that we discuss, we present the **implied risk premia** and the **implied Sharpe Ratios** of the individual assets. Instead of using actual risk premia and the variance-covariance matrix to calculate the maximum Sharpe Ratio portfolio (MSRP), we assume that the portfolio at hand actually ***is*** the MSRP. Together with the variance-covariance matrix of excess returns this allows us to derive the "imputed" risk premia (pioneered by Sharpe [1974]); together with the actual (historical) asset standard deviations, we can then compute the implied Sharpe Ratios. Hence, given a particular portfolio, these implied risk premia (or implied Sharpe Ratios) would make this portfolio the maximum Sharpe Ratio portfolio.

The process of calculating implied risk premia is called **"reverse portfolio optimization"**; for details, see the Technical Appendix. Reverse optimization is relevant when there is uncertainty about ex-ante risk premia. After all, since the MSRP is the tangency portfolio to the mean-variance efficient frontier without including risk free borrowing and lending, this portfolio is very sensitive to the input risk premia. Slight differences in these inputs can result in very different (and sometimes “unrealistic” or extreme and hence unacceptable) portfolios. At the same time, estimating ex-ante risk premia is a very difficult task. Reverse optimization can help since the assets’ implied risk premia serve as a sensible starting point. Depending on the confidence placed in one’s ex-ante views, one can next adjust the implied risk premia accordingly. After this two-stage process, the resulting portfolio is closer to the original portfolio and less extreme. This two-stage portfolio optimization process is proposed by Black & Litterman [1992].

Table 5 : Implied risk premia and implied Sharpe Ratios within market cap portfolio

	Eq	**Tsies**	**USIG**	**USHY**
implied risk premium	9.09	-0.18	2.06	5.11
implied Sharpe Ratio	0.54	-0.04	0.32	0.45

Table 5 presents the implied risk premia and the implied Sharpe Ratios of the market cap portfolio. For Equities, the implied risk premium is about twice as large as the historical risk premium. For IG, the implied risk premium is less than half of the historical risk premium. So when the market cap portfolio would be the MSRP, Equities would have to offer a risk premium of 9% and IG of 2%. Conversely, when we would feel confident in extending the historical risk premia to the future, this implies that we should increase the weight of IG and lower the weight of Equities in order to increase the Sharpe Ratio of the market cap portfolio. For Tsies, the implied risk premium (and hence the implied Sharpe Ratio) is even slightly negative. This reflects the role of Tsies as a hedge in the market cap portfolio. Because of the negative correlation of Tsies with Equities (and HY), their 30% weight in the market cap portfolio would be justified even when their risk premium would be zero.

Notation

We use fairly conventional notation. We denote individual asset standard deviations or volatilities by σ_i. The portfolio volatility is σ_p. The beta of asset i with respect to portfolio p is β_{ip} and its correlation with the portfolio is denoted as ρ_{ip}. The portfolio weight of asset i is denoted as w_i. Where deemed necessary, technical details are mentioned in the main text. For the quant minded, the Technical Appendix contains a general background and additional derivations.

3. 1/N or equal-weighting

Main reference :

- DeMiguel, Garlappi & Uppal [2009] "Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?"

Recipe :

- In equally-weighted portfolios, each asset is assigned a weight of 1/N.
  In our example, each asset class gets a weight of 25% in the portfolio. Since we maintain these weights over time, the 1/N portfolio is rebalanced monthly.

Characteristics :

- 1/N avoids concentrated positions -- in terms of **money allocation** !
- Within equities, 1/N implies an exposure to the **small-cap anomaly**. The market cap portfolio is tilted towards large cap stocks. The 1/N portfolio is tilted towards small cap stocks and will hence capture a size premium.
- 1/N implies a **disciplined and periodical rebalancing** of positions. By definition, the market cap portfolio is a buy-and-hold portfolio. The 1/N portfolio, in contrast, implies a "volatility pumping" effect : in order to maintain the 1/N allocation, one has to buy (sell) out- (under-) performing assets. This is effectively a "buy low, sell high" strategy, which profits from reversals. Depending on the revision period, the rebalancing process implies portfolio turnover with the associated transaction cost and exposure to potential illiquidity (since even the smallest market cap assets get a weight of 1/N).
- **Estimation risk :** in Bayesian terms, the 1/N portfolio is the "uninformed prior" : the naively diversified portfolio that is optimal when one has no information to discriminate between the attractiveness of assets.
- When all assets have the same volatility and when all pairwise correlations are the same, then the 1/N portfolio is the **MVP**. In this case, the MVP also coincides with the **ERCP**. See below.
- 1/N indices are published by MSCI and S&P, among others.

From Table 2 we see that the 1/N portfolio has a higher historical risk premium and a lower risk than the market cap portfolio. This stems mainly from underweighting Equities (with a lower Sharpe Ratio) and overweighting HY (with a higher Sharpe Ratio).

Table 6 shows the 1/N portfolio statistics. It clearly shows that equal money allocation is very different from equal risk contributions. Notably Tsies act as a strong diversifier (negative correlation with Equities and HY) and show (virtually) zero risk contribution. Still, Equity risk dominates in the 1/N portfolio, accounting for 50% of the portfolio volatility. For Equities, the implied risk premium is 10.56% p.a. (which given historical volatility implies a Sharpe Ratio of 0.63). When one believes that the *ex ante* equity risk premium is below 10.56%, the weight of Equities should be lowered in order to improve the risk-adjusted portfolio performance. When one believes that the *ex ante* bond risk premium is above 2 bps p.a., the weight of Tsies should be increased. Equivalent reasoning applies to IG and HY.

Table 6 : Risk attribution with respect to 1/N portfolio, and implied risk premia and SRs.

	Eq	Tsies	IG	HY	sum
weight	25%	25%	25%	25%	
beta	1.99	0.00	0.62	1.39	
% risk contribution	50%	0%	16%	35%	100%
implied risk premium	10.56	0.02	3.30	7.38	
implied Sharpe Ratio	0.63	0.00	0.51	0.66	

4. Maximum Diversification Portfolio MDP

Main reference :

- The MDP is proposed by Choueifaty & Coignard [2008] "Toward Maximum Diversification"

Recipe :

- The weights of the MDP are obtained by maximizing the "diversification ratio", which is defined as the ratio of weighted volatilities and portfolio volatility :

$$(1) \qquad \max_{\{w\}} \frac{\sum_i w_i \sigma_i}{\sigma_p}$$

For obtaining insight into this ratio, note that the portfolio volatility can be written as the weighted sum of the product of each asset's individual volatility and its correlation with the portfolio. Hence, we can rewrite the diversification ratio as :

$$(2) \qquad \max_{\{w\}} \frac{\sum_i w_i \sigma_i}{\sum_i w_i \sigma_i \rho_{ip}}$$

This expression reveals that the diversification ratio compares (i) the portfolio volatility when ignoring correlations in the numerator, with (ii) the actual portfolio volatility when taking into account correlation (and hence diversification) in the denominator. Imperfect (<1) correlations increase the diversification ratio above unity.

Characteristics :

- It can be shown that for the MDP it holds that (see Choueifaty & Coignard [2008]) :

$$(3) \qquad \frac{1}{\sigma_i}\frac{\partial \sigma_p}{\partial w_i} = \frac{1}{\sigma_j}\frac{\partial \sigma_p}{\partial w_j}$$

where $\partial \sigma_p / \partial w_i$ is the marginal contribution of asset i to portfolio volatility. By definition, within the global Minimum Variance Portfolio, all assets' marginal risk contributions are equal, see section 5. It follows that for equal volatilities, $\sigma_i = \sigma_j$, the MPD coincides with the global **MVP**.

- From (3) it also follows that when risk premia $\{\overline{r}_{if}\}$ are proportional to volatilities $\{\sigma_i\}$, thus implying that all assets have the same Sharpe Ratio, then the MDP is the **MSRP**. After all, in the MSRP the assets' marginal contributions to the portfolio risk premium are proportional to the assets' marginal contributions to portfolio volatility, implying :

$$(4) \qquad \frac{1}{\overline{r}_{if}}\frac{\partial \sigma_p}{\partial w_i} = \frac{1}{\overline{r}_{jf}}\frac{\partial \sigma_p}{\partial w_j} \quad \Leftrightarrow \quad \frac{\overline{r}_{if}}{\beta_{ip}} = \frac{\overline{r}_{jf}}{\beta_{jp}}$$

(see section 8).

- Choueifaty & Coignard [2008] also show that each asset has the **same correlation** with the MDP.
- FTSE publishes the FTSE TOBAM Maximum Diversification Index Series.

Evaluation :

- **Why** should one maximize this specific diversification ratio ? After all, there are many definitions of "diversified" !
- The diversification ratio is a **differential** diversification measure. It applies with respect to the specific portfolio at hand. It is no absolute diversification measure from which we can read the degree of diversification; we cannot compare the diversification ratios of two different portfolios to infer which portfolio is more diversified than the other.
- The MDP is **not unique** and may be very **concentrated** in weights (money allocation) or in risk and loss contributions (risk allocations). Indeed, in our example IG carries zero weight in the MDP, see Table 7.

Table 7 : Risk attribution with respect to MDP, and implied risk premia and SRs.

	Eq	**Tsies**	**IG**	**HY**	**sum**
weight	16%	73%	0%	11%	
beta	2.22	0.65	1.20	1.49	
% risk contribution	36%	48%	0%	16%	100%
implied risk premium	9.94	2.93	5.39	6.68	
implied Sharpe Ratio	0.59	0.59	0.84	0.59	

Tsies have the highest weight in the MDP; the money allocation of 73% here implies that Tsies account for about 50% of the portfolio risk. This can hardly be termed a "diversified portfolio"…

- Table 7 also shows that the implied Sharpe Ratios of the three portfolio components equals 0.59. This confirms that when Sharpe Ratios **of the portfolio constituents** are the same, then the MDP is the **MSRP**. Note that this only applies to assets comprised in the MDP; by construction, the composition of the MDP does not depend on risk premia or Sharpe Ratios.

- The portfolio statistics are depicted in Table 8. For the historical inputs, the MDP beats the market cap and 1/N portfolios in risk-adjusted performance. This is due to the large overweight of Tsies which over the past decade showed the highest Sharpe Ratio.

Table 8 : Comparative portfolio statistics.

Portfolio stats	Cap Wtd	1/N	MDP
avge	4.71	5.31	4.47
stdev	8.24	7.30	4.26
SR	0.57	0.73	1.05

- Finally, note that we use the full historical sample to calculate the weights of the MDP. In practical applications, one would use trailing historical windows (avoiding a look-ahead bias) to re-calculate the weights. In this way, the out-of-sample properties of the MDP can be evaluated.

5. Minimum Variance Portfolio MVP

Main references :

- Haugen & Baker [1991] “The Efficient Market Inefficiency of Capitalization-Weighted Stock Portfolios”,
  show that market cap weighted portfolios are inefficient (sub-optimal) when there are market frictions and highlight the high relative performance of low volatility portfolios
- Clarke, DeSilva & Thorley [2006] “Minimum Variance Portfolios in the US Equity Market”, extend Haugen & Baker’s empirical research
- Blitz & van Vliet [2007] “The Volatility Effect : Lower Risk Without Lower Return”,
  revive the interest in the low volatility anomaly and provide possible explanations (behavioural biases, leverage restrictions, and delegated portfolio management and benchmarking)
- Scherer [2011] “A Note on the Returns From Minimum Variance Investing”,
  links the returns on the MVP to factor premia

Recipe :

- Choose the portfolio weights to minimize portfolio variance :

(5) $$\max_{\{w\}} \sigma_p^2 = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij}$$

- The optimal portfolio is characterized by equal marginal contributions to portfolio risk :
$$\frac{\partial \sigma_p}{\partial w_i} = \frac{\partial \sigma_p}{\partial w_j}$$

Characteristics :

- Note that marginal risk contributions are given by $\frac{\partial \sigma_p}{\partial w_i} = \frac{\sigma_{i,p}}{\sigma_p} = \beta_{ip} \cdot \sigma_p$, so all asset **betas** with respect to the MVP are identical.
- Since an asset's risk contribution equals $w_i \frac{\partial \sigma_p}{\partial w_i} \sim w_i$, risk contribution is proportional to the investment weight, so **risk allocation equals money allocation**.
- When all assets have the same volatility and when all pairwise correlations are the same, then the MVP is the **1/N portfolio**. After all : it pays to diversify over the assets but in the portfolio context, all assets are perfect substitutes.
- the MVP is the **MSRP** when all assets have the same risk premium, $\overline{r_{if}} = \overline{r}$ (implying that all Sharpe Ratios SR_i are proportional to $1/\sigma_i$). After all, in that case we have (cf. eq.(4)) :

$$\text{(6)} \qquad \frac{1}{\overline{r}} \frac{\partial \sigma_p}{\partial w_i} = \frac{1}{\overline{r}} \frac{\partial \sigma_p}{\partial w_j}$$

Evaluation :

- The MVP favours low volatility assets and low beta assets and hence benefits from the low volatility anomaly. The **MSCI Minimum Variance Index** and the **S&P Low Volatility** Index are examples of low risk portfolios that are designed to benefit from this anomaly.
  For more information on the low volatility anomaly, see Blitz & van Vliet [2007].
- Several studies have documented that MVPs also pick up other priced anomalies. Clarke, DeSilva & Thorley [2006] find that, in general, the MVP has a substantially higher **value (B/P) exposure** than the market (since value stocks tend to have low volatilities), which explains at least part of its higher average realized return. Scherer [2011] shows that the MVP loads significantly on the Fama-French factors (large size and high value) but also finds that MVPs have a negative beta bias (favour **low beta** assets) and favour assets with **low residual volatility**. The latter effects crowd out the Fama-French factors in the sense that low beta and low residual volatility alone can explain more of the variation in the MVP's excess returns than the Fama-French factors. This leads Scherer to conclude that low beta and low residual volatility is a more efficient and effective way to capture the low volatility anomaly than minimum variance.
- When time passes and the MVP is re-optimized, one will need to apply constraints on turn-over in order to mitigate transactions costs. However, turnover constraints make the MVP a **path dependent** strategy.
- The MVP is a **concentrated** portfolio. Assets with low volatility and/or low correlations with other assets will carry a large weight. Conversely, assets with high volatility and/or high correlations with other assets will carry a small or even negative weight; when excluding short positions, these assets will not appear in the MVP. This is illustrated in Table 9 : IG is not included in the long-only MVP.

Table 9 : Risk attribution with respect to MVP, and implied risk premia and SRs.

	Eq	Tsies	IG	HY	sum
weight	8%	80%	0%	12%	
beta	1.00	1.00	1.31	1.00	
% risk contribution	8%	80%	0%	12%	100%
implied risk premium	4.44	4.44	5.84	4.44	
implied Sharpe Ratio	0.27	0.90	0.91	0.39	

Table 9 confirms that when the **assets comprised in the MVP** have identical risk premia, then the MVP is the **MSRP**. Note again that this only applies to assets that are comprised in the MVP in the first place.

- Table 9 also confirms marginal risk contributions of MVP constituents are identical (all betas equal unity) and that money allocation equals risk allocation in a MVP.
- Table 10 shows the portfolio statistics. The MVP risk premium is about the same as the MDP's risk premium, but its volatility is lower, thus yielding a higher Sharpe Ratio. This lower volatility is achieved by overweighting Tsies at 80%, supplemented by positions in Equities and HY which are negatively correlated with Tsies.

Table 10 : Comparative portfolio statistics.

	Cap Wtd	1/N	MDP	MVP
avge	4.71	5.31	4.47	4.44
stdev	8.24	7.30	4.26	3.99
SR	0.57	0.73	1.05	1.11

- Last but not least, the quadratic optimization underlying the MVP has the property of being "error maximizing", see Michaud [1989]. This implies that the composition of the MVP is very sensitive to slight differences in variances and covariances. When (part of) these differences are not real but due to sampling error, this will propagate into portfolio composition.
- Again, note that we use the full historical sample to calculate the weights of the MVP.

6. Equal Risk Contribution portfolio ERCP – full Risk Parity

Main references :

- Qian [2005], "Risk Parity Portfolios : Efficient Portfolios Through True Diversification", this is the seminal paper on risk parity
- Qian [2006], "On the Financial Interpretation of Risk Contribution; Risk Budgets Do Add Up", this paper is on the linear decomposition of risk

- Hallerbach [2003], “Decomposing Portfolio Value-at-Risk, A General Analysis”, extends risk decomposition to Value-at-Risk and shows how to decompose risk in parametric and non-parametric (simulation) settings
- Maillard ea [2010],”The Properties of Equally Weighted Risk Contribution Portfolios”, discusses the theoretical properties of risk parity portfolios and provides a comparison with other risk control techniques
- Lee [2011] “Risk-Based Asset Allocation : A New Answer to an Old Question ?”, provides a good discussion of risk control techniques, with especially a critical evaluation of Risk Parity (see also section 10)
- Asness, Frazzini & Pedersen [2012] “Leverage Aversion and Risk Parity”, document the empirical outperformance of a risk parity strategy over a market cap weighted portfolio and refer to the leverage aversion effect to explain this outperformance
- Anderson, Bianchi & Goldberg [2012] “Will My Risk Parity Strategy Outperform?”, critically review and refute the empirical evidence provided by Asness, Frazzini & Pedersen [2012]

Recipe :

- The ERCP rests on the premise that no asset should dominate the portfolio risk profile. Consequently, all assets’ contributions to portfolio risk are equalized. The contribution of an asset to portfolio risk equals its investment weight multiplied with its marginal contribution to portfolio risk. An asset’s marginal contribution to portfolio risk equals its beta with respect to the portfolio. Hence, the weights of the ERCP satisfy :

$$(7) \qquad w_i \frac{\partial \sigma_p}{\partial w_i} = \frac{\partial \sigma_p}{\partial w_j} w_j \quad \Leftrightarrow \quad w_i \beta_{ip} = w_j \beta_{jp}$$

Hence, the weights in the ERCP are proportional to the inverse of the corresponding betas :

$$(8) \qquad w_i^{ERC} \sim \frac{1}{\beta_{ip}}$$

- Since by definition the contribution of each asset to portfolio risk must equal 1/N, the composition of the ERCP can easily be calculated in **Excel** by requiring that for each asset $w_i \beta_{ip} = 1/N$.

Characteristics :

- The ERCP is the **1/N** portfolio when all assets have the same volatility σ and when all pairwise correlations are uniform at ρ . After all, in that case eq.(7) implies that $w_i \sigma \rho = w_j \sigma \rho$, which is satisfied for $w_i = w_j = 1/N$.
- The ERCP is the **MDP** when all correlations are uniform : $\rho_{ip} = \rho_{jp}$.
- The ERCP is the **MVP** when correlations are uniform (pairwise equal) ànd at their theoretically lowest level of $\rho = -1/(N-1)$. See Maillard ea [2010].

- The ERCP is the **MSRP** when all correlations are uniform ànd all assets have the same Sharpe Ratio.
  See Maillard ea [2010].
- When there are only two assets, the ERCP equals the IVP (see section 7).

Evaluation :

- "Risk" is usually equated with standard deviation of return (volatility), but in principle any other risk measure can be chosen as long as the risk measure is linearly homogeneous in the portfolio weights. This means that when multiplying all investment weights with a constant c, the risk measure is also multiplied by the same constant c. Portfolio loss, Value-at-Risk (VaR) and Conditional VaR (or Expected Tail Loss) satisfy this property. See Hallerbach [2003], e.g.
- Since we can rewrite beta as the product of (1) the correlation with the portfolio and (2) the quotient of the asset and portfolio volatility, so $\beta_{ip} = \rho_{ip}\sigma_{if} / \sigma_{pf}$, eq.(8) implies that ERCPs favour assets with low levels of volatility and low correlations with other assets (hence : **"portfolio diversifiers"**)
- The portfolio statistics are depicted in Table 11 :

Table 11 : Comparative portfolio statistics.

	Cap Wtd	1/N	MDP	MVP	ERCP (4)	ERCP (3)
avge	4.71	5.31	4.47	4.44	4.79	4.47
stdev	8.24	7.30	4.26	3.99	4.77	4.60
SR	0.57	0.73	1.05	1.11	1.00	0.97

  ERCP(4) is on the basis of the 4 original assets, in ERCP(3), IG and HY are combined into one asset class. The table shows that the ERCPs had about half the risk of the market cap portfolio at comparable levels of average return, yielding almost double Sharpe Ratios. This is due to overweighting Tsies and underweighting Equities (see Table 12).
- The ERCP is perfectly **diversified** in terms of risk (loss) contributions.
- The ERCP is **less concentrated** than the MVP and the MDP, and it contains all *N* assets.
- The ERCP is more **robust**, i.e. less error maximizing, than the MVP. The intuitive reason is that the MVP is found by means of **optimization**, i.e. by equating marginal risk contributions, whereas the ERCP is found by a **restriction** on the product of weights and marginal risk contributions.
- It can be shown that $\sigma_{MVP} \leq \sigma_{ERC} \leq \sigma_{1/N}$, where the MVP is error maximizing and the 1/N portfolio focuses on money allocation, not risk allocation. Hence, the ex-ante volatility of the ERCP is between the lowest level (from the MVP) and the volatility of the naively diversified 1/N portfolio.
  See Maillard ea [2010] for details.
- Calculating the ERCP is a daunting task when the number of assets is very large. A solution would be to resort to a hierarchical procedure in which risk parity is first applied within groups

(sectors, countries,…) and next across groups. However, pre-grouping directly influences the ERCP, see below.

- Table 12, Panel A, shows the composition of the ERCP. Note the large 52% weight of Tsies, this is due to both their low volatility and their negative correlation with Equities and HY. The high volatility of Equities implies a lower than 25% weight. The implied risk premia and Sharpe Ratios can be interpreted as before.

Table 12 : Risk attribution with respect to ERCP, and implied risk premia and SRs.

Panel A : ERC (4)	**Eq**	**Tsies**	**IG**	**HY**	**sum**
weight	12%	52%	21%	15%	
beta	2.06	0.48	1.21	1.68	
% risk contribution	25%	25%	25%	25%	100%
implied risk premium	9.86	2.29	5.77	8.02	
implied Sharpe Ratio	0.59	0.46	0.90	0.71	
Panel B : ERC (3)	**Eq**	**Tsies**	**IG+HY**		
weight	16%	57%	26%		
beta	2.06	0.58	1.26		
% risk contribution	33%	33%	33%		100%
implied risk premium	9.19	2.59	5.64		
implied Sharpe Ratio	0.55	0.53	0.84		

- The composition of the ERCP depends on choosing the number of assets *N* and hence on any **pre-grouping** of assets (see Lee [2011]). For example, when aggregating IG and HY into a single credits sub-portfolio, ERCP(3), the risk allocations shift from 25% to 33%; see Table 12, Panel B. In Table 11 we see that, in this particular example, combining IG and HY has almost no historical performance consequences.
- **Leverage** is needed to boost the low risk and return of RP in order to match any risk budgets or return targets.
- Again, note that we use the full historical sample to calculate the weights of the ERCP. In practice, one would sequentially derive ERCPs over rolling data windows. In back-tests, one should avoid any look-ahead biases when implementing leverage and rebalancing.
- In their empirical study, Asness, Frazzini & Pedersen [2012] illustrate the **historical outperformance** of ERCPs (or IVPs since they consider only two asset classes, US equity and bonds) over a market cap weighted portfolio over the period 1926-2010. As an explanation they raise **leverage aversion** as the driving force behind the performance of ERCPs. This mechanism works as follows. (Some) investors are averse (or restricted) to applying leverage and they bid up the prices of high risk / high beta assets in order to fill their risk budget. As a consequence, the risk premium offered on high risk assets is reduced. Low beta (risk) assets offer higher risk-adjusted returns, and high beta (risk) assets offer lower risk-adjusted returns. A less than average leverage-averse / -constrained investor can benefit by overweighting low beta (risk) assets and

underweighting high beta (risk) assets. Leverage is applied to fill the risk budget or to attain a targeted risk level. In addition to leverage aversion, the **“lottery ticket effect”** may be at work, in which investors with a propensity to “gamble” overbid for high risk assets, thus reducing their risk premium. Finally, delegated portfolio management, centered around benchmarked portfolios, implies that low (high) risk stocks have large (small) tracking error. As argued by Blitz & van Vliet [2007], this introduces **the low volatility anomaly**, implying a flat or negative risk-return trade-off. Since low volatility assets outperform and ERCPs overweight low risk assets, this may explain their outperformance.

- Anderson, Bianchi & Goldberg [2012] raise some serious back test issues in the research by Asness, Frazzini & Pedersen [2012]. First of all, they note that the outperformance of the ERCP is not uniform over sub-periods. Secondly, they show that market frictions (borrowing costs and turn-over induced trading costs) eat into performance. Thirdly, they argue that Asness, Frazzini & Pedersen’s [2012] risk parity strategy is not an investable strategy since it uses unconditional leverage : they use a constant scale factor, computed from the full 1926-2010 period, to match the volatilities of the levered risk parity strategy and the market cap portfolio. Hence, their empirical set-up suffers from a look-ahead bias. Anderson, Bianchi & Goldberg [2012], in contrast, use conditional leverage where at each rebalancing date the volatility scale factor is derived from past 3Y trailing windows. They show that implementing conditional leverage halves the cumulative total return of the risk parity strategy as reported by Asness, Frazzini & Pedersen [2012]. Realistic borrowing costs and trading costs further reduce the cumulative total return of the risk parity strategy. In all, these realistic adjustments make the performance difference between the risk parity strategy and the market cap portfolio disappear…

7. Inverse Volatility Portfolio IVP – naive Risk Parity

Main reference :

- Maillard ea [2010],”The Properties of Equally Weighted Risk Contribution Portfolios”, they discuss IVP next to ERCP, although volatility weighting (or “normalization”) has been applied for long by practitioners to improve cross-asset comparability and to reduce portfolio or strategy risk. (This may be inspired by statistics, where inverse variance weighting is used to minimize the variance of the sum of two or more random variables.)

Recipe :

- Set each weight proportional to the stand-alone volatility of the corresponding asset and normalize so that the weights sum to unity. This volatility-weighting in the cross-section yields :

$$w_i = \frac{\frac{1}{\sigma_i}}{\sum_j \frac{1}{\sigma_j}} \tag{9}$$

- The IVP is equivalent to the **ERCP** when there are only two assets. (In the two-asset case, the correlation is irrelevant.)

- The IVP is equivalent to the **ERCP** when correlations are uniform (or zero). Neglecting correlation information makes IVP a “naïve” risk parity strategy.

Characteristics :

- When correlations are uniform (or zero), the IVP is the **ECRP**. (In that case, all comments made for ERCPs also apply for IVPs). When everything else is equal, then compared to the IVP, the ERCP will be tilted towards low correlated assets.
- When correlations ànd volatilities are uniform, the IVP is the **1/N** portfolio.
- The **S&P Low Volatility Index** is composed of the 100 stocks from the S&P500 Index with the lowest (252 days past) volatility, where each stock is weighted with its inverse volatility.
- The **MSCI Risk Weighted Indices** use inverse variance (and not volatility) to weight constituents. Inverse variance weighting yields the **MVP** when all correlations are uniform (or zero).

Evaluation :

- Except for the impact of (markedly different) correlations, IVPs are quite similar to ERCPs. As shown in Table 13, the IVP assigns more weight to IG (was 21%) and less weight to Tsies (was 52%). The latter can be explained because the IVP ignores the negative correlation with Equities and HY. This shift in weights translates into less balanced risk contributions.

Table 13 : Risk attribution with respect to IVP, and implied risk premia and SRs.

	Eq	Tsies	IG	HY	sum
weight	12%	40%	31%	18%	
beta	2.01	0.34	1.09	1.67	
% risk contribution	24%	13%	34%	29%	100%
implied risk premium	9.99	1.67	5.44	8.32	
implied Sharpe Ratio	0.60	0.34	0.85	0.74	

- Table 14 shows that the IVP has somewhat higher volatility and average return than the ERCP. This combined effect is due to the lower weight of Tsies (which have the lowest average return, the lowest volatility, and negative correlations with Equities and HY).

Table 14 : Comparative portfolio statistics.

	Cap Wtd	1/N	MDP	MVP	ERCP	IVP
avge	4.71	5.31	4.47	4.44	4.79	4.97
stdev	8.24	7.30	4.26	3.99	4.77	5.28
SR	0.57	0.73	1.05	1.11	1.00	0.94

8. Maximum Sharpe Ratio Portfolio MSRP

Main references :

- For a discussion of the Sharpe ratio, see Sharpe [1994].
- For mean-variance portfolio theory and for finding the MSRP, we refer to standard investment texts.

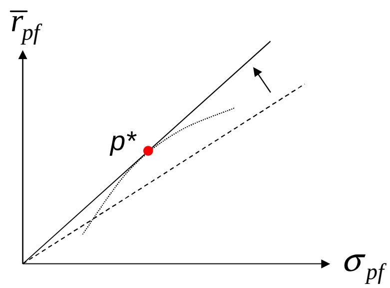
Recipe :

- Choose the portfolio weights to maximize the Sharpe Ratio :

(10) $$\max_{\{w\}} SR_p = \frac{\overline{r}_{pf}}{\sigma_{pf}}$$

  This can be accomplished by quadratic optimization, or in Excel by first defining an extra column with portfolio returns given an array of weights and next to maximize the Sharpe Ratio of this portfolio returns series.

- Hence, in the familiar excess return-risk graph, we should maximize the slope of the ray emanating from the origin, as shown in the figure below :



Characteristics :

- Within the MSRP, the ratios of marginal contributions to risk and return are constant. Since an asset's marginal contribution to the portfolio risk premium equals the asset's risk premium, $\overline{r}_{if}$, and since this asset's marginal contribution to portfolio risk is its beta, β_{ip} , we require : $\overline{r}_{if} / \beta_{ip} = \overline{r}_{jf} / \beta_{jp} = \overline{r}_{pf}$ (where the last equality follows from the fact that the portfolio beta equals unity). Note that $\overline{r}_{if} / \beta_{ip}$ is the Treynor [1966] risk-adjusted performance ratio. Hence, for each asset within the MSRP, the risk premium should be equal to the product of its beta with respect to the MSRP and the risk premium of the MSRP :

(11) $$\overline{r}_{if} = \beta_{ip}\overline{r}_{pf} .$$

  This is the first-order condition of mean-variance optimality. (When invoking market equilibrium, this becomes the familiar "Security Market Line" of the CAPM.)

- Since we can rewrite the beta as the product of (1) the correlation with the portfolio and (2) the quotient of the asset and portfolio volatility, $\beta_{ip} = \rho_{ip}\sigma_{if} / \sigma_{pf}$, it follows that in the MSRP the stand-alone asset Sharpe Ratios and the portfolio's Sharpe ratio are related by : $SR_i = \rho_{ip} SR_p$. When $SR_i > \rho_{ip} SR_p$, we can increase the Sharpe Ratio of the portfolio by increasing the weight of (or adding) the asset to the portfolio *p*.

- Without any additional constraints, the long-only MSRP can be a very concentrated portfolio.
- When all volatilities, correlations and risk premia are the same, then the MSRP is the **1/N portfolio** (which then also coincides with the **ERCP** and the **MVP**). After all, diversification lowers risk but in the portfolio context all assets are perfect substitutes. It is not possible to lower portfolio risk or increase the portfolio risk premium by changing the weights. Hence, we end up with the equally-weighted portfolio.

Evaluation :

- The MSRP has the maximum Sharpe Ratio, see Table 15. This is so by construction, since we optimized the Sharpe Ratio over the full historical sample period (in-sample). In practice, one would sequentially derive the *ex ante* MSRP from trailing data windows. Whether the MSRP indeed delivers the maximum Sharpe Ratio *ex post* depends on the quality of the inputs, especially the risk premia.

Table 15 : Comparative portfolio statistics.

	Cap Wtd	1/N	MDP	MVP	ERCP	IVP	MSRP
avge	4.71	5.31	4.47	4.44	4.79	4.97	4.98
stdev	8.24	7.30	4.26	3.99	4.77	5.28	4.20
SR	0.57	0.73	1.05	1.11	1.00	0.94	1.19

- In our example, the MSRP is indeed a concentrated portfolio, containing mostly Tsies supplemented with HY, see Table 16. Tsies dominate because of their low volatility and negative correlation with HY. The smaller than unity beta of Tsies reveal that Tsies are included as a diversifier; the larger than unity beta of HY shows that HY is included because of its (highest) average return). Slight changes in the risk premia of Tsies and HY will change the composition of the MSRP markedly.

Table 16 : Risk attribution with respect to MSRP, and implied risk premia and SRs.

	Eq	Tsies	IG	HY	sum
weight	0%	72%	0%	28%	
beta	1.10	0.79	1.36	1.53	
% risk contribution	0%	57%	0%	43%	100%
implied risk premium	5.49	3.93	6.77	7.65	
implied Sharpe Ratio	0.33	0.80	1.05	0.68	

9. Volatility weighting over time

The risk control strategies as discussed before focus on risk in the **cross-section**, i.e. over portfolio constituents. Risk control at each point in time will also affect the portfolio's risk level (or more generally, its return distribution) **over time**. Volatility weighting over time, and specifically v**olatility targeting**, is designed to explicitly control the portfolio risk level over time.

Main references :

- Kirby & Ostdiek [2012] "It's All in the Timing : Simple Active Portfolio Strategies that Outperform Naive Diversification",
  volatility weighting over time is quite widespread in practice, but this paper documents the empirical finding that volatility weighting improves the Sharpe Ratio
- Hallerbach [2012], "A Proof of the optimality of volatility weighting over time",
  the title is self-explanatory …. The result holds under mild assumptions.

Recipe :

- Set the risky portfolio's target volatility level *V*
- At the start of each period t, take a position w in the risky portfolio and (1-w) in the risk free asset :

(12) $$w_t \cdot \tilde{r}_{pt} + (1 - w_t) \cdot r_{ft} = w_t \cdot \tilde{r}_{pft} + r_{ft}$$

- Estimate the volatility of the risky portfolio for period t : $\hat{\sigma}_t$.
  For example by using an adaptive Exponentially-Weighted Moving Average (EWMA) volatility process.
- Rescale the exposure to the risky portfolio to the target volatility level V : $w_t = \dfrac{V}{\hat{\sigma}_t}$.
  According to (12), this implies adding a cash position or borrowing (when allowed) at the suitable borrowing rate, subject to a leverage constraint.
- Apply the leverage constraint. When the volatility target V is high or when the forecasted volatility is low, cap the implied borrowing by setting $w_t \leq L$, where the maximum leverage ratio satisfies $L \geq 1$. When $L = 1$, no borrowing is allowed.

Characteristics :

- Volatility weighting and volatility targeting accomplish **volatility smoothing** over time.
- Volatility smoothing mitigates the volatility of the portfolio volatility over time. It can be shown that the lower the fluctuations of the temporal ("instantaneous") portfolio volatility **within** some time period, the lower the aggregate volatility **over** the whole time period. For details, see Hallerbach [2012].
- Note that volatility smoothing is different from **return smoothing**. Return smoothing aims at achieving a lower aggregate level of return volatility (and not a lower volatility of the volatility over time). Return smoothing thus implies less "variance slippage" in compounded returns. This variance slippage refers to the difference between the geometric mean and the arithmetic mean.

As an approximation, we have
geometric mean ≈ arithmetic mean – ½ variance. Lowering the return variance by return smoothing thus increases the geometric mean of returns, cet. par.

- Naive Risk Parity or the **IVP**, i.e. vol weighting in XS, already establishes some volatility weighting in TS.
- Risk targeting or risk control indices have been introduced by S&P, MSCI, FTSE, DJ, and EURO STOXX.

Evaluation – or : Why would volatility targeting work ?

- First of all, depending on the quality of our volatility forecasts, we should be able to target a portfolio's volatility to some degree over time.
- In addition, it can be shown that this volatility smoothing increases the Sharpe Ratio or Information Ratio of the portfolio, cet. par. (for details, see Hallerbach [2012]).
- Furthermore, the (risk-adjusted) return of a volatility targeted portfolio benefits from an additional timing effect, due to the so-called **asymmetric volatility phenomenon**. The asymmetric volatility phenomenon is a stylized fact that is observed for most financial markets. In general, returns tend to be negatively correlated with the volatility of subsequent returns. More specifically, surges in financial market volatility are mostly associated with negative returns. The volatility feedback mechanism is that higher expected volatility translates into a higher risk premium and hence lower realized returns. Hence, under asymmetric volatility, there is a **timing effect** (in addition to the smoothing of volatility) that will boost performance. After all, a volatility-weighting strategy takes large positions when volatility is low (and returns are high) and small positions when volatility is high (and returns are low).
- As a cautionary (and perhaps superfluous) note, we stress that implementing a volatility-weighted strategy calls for a strict risk-budgeting and risk-monitoring process. In particular, one may want to set limits to the maximum position size in order to mitigate the risk of **blow-ups** when the contemporaneous volatility is relatively low.

10. Evaluation

Main references :

- Inker [2011], "The Dangers of Risk Parity"
- Lee [2011], "Risk-Based Asset Allocation : A New Answer to an Old Question ?"
- Leote de Carvalho, Lu & Moulin [2012], "Demystifying Equity Risk-Based Strategies : A Simple Alpha plus Beta Description"
- Goldberg & Mahmoud [2013], "Risk Without Return"

Using risk control techniques (and especially Risk Parity) as full-fledged investment criteria is sometimes coined the "new paradigm" in investing. Indeed, conventional 60/40 portfolios or MSRPs are concentrated in risks and fail to offer diversification against losses. Risk Control strategies, and Risk Parity in particular, can produce balanced portfolios and can offer various degrees of diversification.

From a risk perspective, these techniques are indeed expected to deliver what they promise. The catch is that risk control portfolios appear to have historically outperformed market cap weighted or mean-variance optimized portfolios. So while ignoring risk premia information, Risk Control strategies seem to offer a better (i.e. more efficient) risk-return trade-off.

What could be the mechanisms behind this “miraculous” performance of Risk Control strategies ? First of all, several studies tune down the apparent outperformance of risk-based strategies by criticizing back-tests, see section 6 and Goldberg & Mahmoud [2013]. Secondly, when the underlying mechanism of outperformance is an implicit exposure to **anomalies or factor premia** such as value, size, low beta or low (residual) volatility (as shown by Leote de Carvalho, Lu & Moulin [2012]), then it makes much more sense to consider these factor exposures explicitly when forming portfolios. **Factor investing** provides much more efficient and effective ways to tailor factor exposures on the portfolio level than applying risk control techniques. After all, in the latter case one has to wait what factor exposures will percolate bottom-up and reveal themselves in the portfolio.

In section 4 we saw that the MDP is the MSRP if all assets have identical Sharpe Ratios. When we add the condition that correlations are uniform across the whole asset universe, then the ERCP is the MSRP (see section 6). So one could use the argument of **estimation risk** to justify the use of risk control techniques : when we do not have information to meaningfully differentiate between assets (same risk-return trade-offs and hence equal Sharpe Ratios, and same correlations) the recipe is to treat all assets as “substitutes”. But this kind of reasoning leads to an inconsistency, as Lee [2011] argues. After all, when all correlations are smaller than unity, combining two assets A and B with equal Sharpe Ratios yields an asset AB which has a higher Sharpe Ratio, thus violating the assumption of equal Sharpe Ratios. Only when all correlations are perfect (=1), the all Sharpe ratios can be equal – but this in turn implies that all assets are *perfect* substitutes (and hence are redundant).

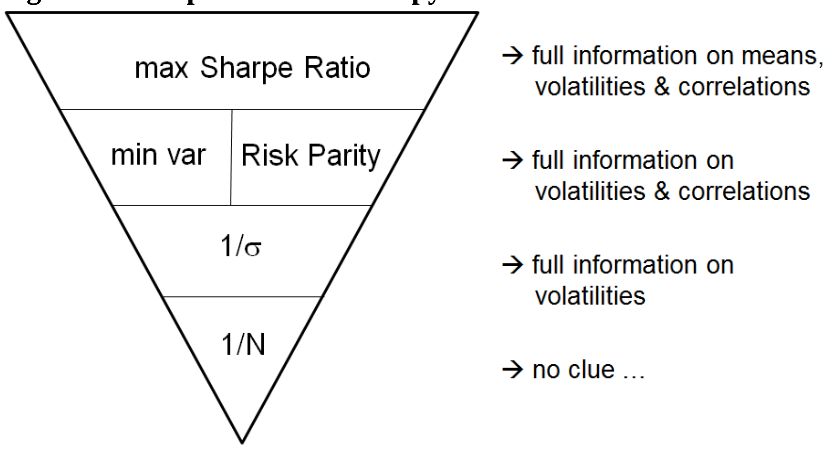
Of course, we do not have to carry this reasoning to the extreme. We do know that mean-variance optimized portfolios are error-maximizing (Michaud [1989]) in the sense that their composition is very sensitive to inputs (especially risk premia). In this context, the adagium “garbage in, garbage out” applies *a fortiori*. As outlined in section 2 and the Technical Appendix, this estimation risk can be tackled by the “Black-Litterman” [1992] approach : start from a Risk Control portfolio, calculate the implied risk premia and next use your views and the confidence you place in these views to (slightly) adjust the optimization inputs. This procedure results in a less extreme and more robust portfolio. A Risk Control portfolio then serves as a **starting point** in the portfolio formation process, and this is quite different from accepting a Risk Control portfolio as a generator of superior risk-adjusted returns *per se*.

Coming to a **conclusion**, we note that the true value of Risk Control strategies is in analyzing and specifying a preferred risk contribution profile within the portfolio. This should be part of any risk budgeting process. Relevant questions are : What are the risk contributions of my portfolio components ?, Is my portfolio properly diversified or are there any hot spots ?, How much confidence do you have in risk premia views in order to shift risk contributions within your portfolio ? Do you fully understand the sources and contributions of risk and return of your portfolio ? And last but not least, risk is a multidimensional concept, so risk analyses should not only focus on volatility (standard deviation) but also take downside risk and event risk in consideration.

Risk Control strategies and Risk Parity are no panacea in reaping outperformance. Under realistic and implementable back-test assumptions, their outperformance can be linked to overweighting asset classes that in the rear view mirror have paired high historical risk premia with low risk levels (as is the case for bonds, e.g.) or to implicit exposures to factor premia (anomalies). Focusing directly on factor exposures provides a more explicit and more efficient and effective way to capture anomalies and earning factor premia.

Still, aside from their value in a risk budgeting context, Rick Control strategies provide a sensible starting point in portfolio optimization when there is considerable uncertainty about the required inputs. To illustrate this point, we introduce our portfolio decision pyramid, see Figure 1.

Figure 1 : The portfolio decision pyramid



- Starting at the bottom of this inverted pyramid, one has no clue about risk and return inputs. The only relevant recipe is then to diversify equally across portfolio constituents, yielding the 1/N portfolio.
- When one can only put reliable trust in volatilities, a portfolio can be formed by applying volatility-weighting, yielding the IVP.
- When one has full risk information (reliable estimates of both volatilities and correlations, so the full covariance matrix is available), the MVP, the MDP or ERCP (Risk Parity portfolios) can be constructed. Of course, one has to take into account the relative shortcomings of these portfolios as noted in the relevant sections above.
- When one trusts all relevant inputs, i.e. the *ex ante* covariance matrix ànd risk premia, then one can build the MSRP.
- When one accepts risk inputs but at the same time acknowledges the estimation risk attached to future risk premia, Risk Control portfolios can serve as a valid starting point in a Black-Litterman [1992] procedure.

11. Technical appendix

Asset (excess) returns

We start with an opportunity set of *N* securities with returns $\tilde{r}_{it}$. Tildes indicate random variables. For notational simplicity, we henceforth ignore the time index *t*.

The risk free rate is denoted by r_f, so the excess returns are $\tilde{r}_i - r_f \equiv \tilde{r}_{if}$.

Portfolio (excess) returns

We consider a portfolio *p* defined by the investment weights $\{w_i\}_{i\in p}$, satisfying full investment $\sum_{i=1}^{N} w_i = 1$ and no short positions : $w_i \geq 0,\ \forall i \in p$.

The portfolio return is given by $\tilde{r}_p = \sum_{i\in p} w_i \tilde{r}_i$. Likewise, the portfolio excess return is given by :

(13) $\tilde{r}_{pf} = \sum_i w_i \, \tilde{r}_{if}$

In Excel this is easily computed by the function **"SUMPRODUCT([w], [r])"**.

Marginal and component contributions to portfolio (excess) return

It follows from eq.(13) that the marginal contribution of asset *i* to portfolio excess return is given by $\tilde{r}_{if}$.

This is the increase in portfolio excess return when the weight of asset *i* is increased marginally.

The component (i.e. full) contribution of asset *i* to portfolio excess return is $w_i \tilde{r}_{if}$. The sum of component contributions to excess return equals the portfolio's excess return, see eq.(13).

Portfolio risk premium

The average portfolio return over the risk free rate, the portfolio risk premium, follows as :

(14) $\overline{r}_{pf} = \sum_{i\in p} w_i \, \overline{r}_{if}$.

The marginal and component contributions of asset *i* to the portfolio risk premium are $\overline{r}_{if}$ and $w_i \overline{r}_{if}$, respectively.

Portfolio variance

The portfolio variance is defined by the double sum :

(15) $\sigma_{pf}^2 = \sum_i \sum_j w_i w_j \sigma_{ij}$.

By definition of the correlation ρ_{ij}, the covariance σ_{ij} can be expressed as $\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$.

Since :

- the variance of a variable is the covariance of that variable with itself, and
- the covariance is a linear operator (the covariance of a weighted sum is the weighted sum of covariances),

we can write the variance of the portfolio excess return as :

(16) $\operatorname{var}\left(\tilde{r}_{pf}\right) \equiv \operatorname{cov}\left(\tilde{r}_{pf}, \tilde{r}_{pf}\right) = \operatorname{cov}\left(\sum_i w_i \tilde{r}_{if}, \tilde{r}_{pf}\right) = \sum_i w_i \operatorname{cov}\left(\tilde{r}_{if}, \tilde{r}_{pf}\right) = \sum_i w_i \sigma_{ipf}$,

where σ_{ipf} is the covariance between the excess returns on asset *i* and the portfolio *p*.

So although the portfolio variance is the quadratic sum of weights and covariances, we can express the portfolio variance as the weighted sum of the covariances of each asset with the portfolio : $\sigma_{pf}^2 = \sum_i w_i \sigma_{ipf}$.

Decomposing portfolio volatility

Dividing the previous expression by the portfolio volatility we get :

(17) $$\sigma_{pf} = \sum_i w_i \frac{\sigma_{ipf}}{\sigma_{pf}} .$$

Indeed, it is not the decomposition of portfolio variance we are looking for, but the **decomposition of portfolio volatility**, as defined by eq.(17). To see why this is true, note that the portfolio volatility is linearly homogeneous in the portfolio weights : multiplying portfolio weights with a constant *k* multiplies the portfolio volatility with the same constant *k*. Euler's theorem then implies that $\sigma_{pf} = \sum_i w_i \frac{\partial \sigma_{pf}}{\partial w_i}$, where it can be checked from (16) that $\frac{\partial \sigma_{pf}}{\partial w_i} = \frac{\sigma_{ipf}}{\sigma_{pf}}$. The term $\frac{\partial \sigma_{pf}}{\partial w_i}$ is the **marginal contribution of asset *i* to portfolio volatility.** The term $w_i \frac{\sigma_{ipf}}{\sigma_{pf}}$ is the **component contribution of asset *i* to portfolio volatility.** The sum of all component contributions to volatility equals total portfolio volatility, see eq.(17). The portfolio volatility is the cake and each component contribution is a separate piece of that cake. Dividing (17) by σ_{pf} yields the relative risk contributions of the assets, summing to 100% :

(18) $$1 = \sum_i w_i \frac{\sigma_{ipf}}{\sigma_{pf}^2} .$$

To gain further insight into this decomposition, consider the OLS regression of asset's *i* excess returns on the portfolio excess returns :

(19) $$\tilde{r}_{if} = \alpha_i + \beta_{ip} \tilde{r}_{pf} + \tilde{\varepsilon}_i .$$

In this regression, the expected (or average) value of the disturbances is zero and the disturbances and the portfolio excess return are uncorrelated, hence $E\left(\tilde{\varepsilon}_i\right) = E\left(\tilde{r}_{pf} \tilde{\varepsilon}_i\right) = 0$. The regression slope or beta is defined as :

(20) $$\beta_{ip} = \frac{\sigma_{ipf}}{\sigma_{pf}^2} = \rho_{ip} \frac{\sigma_{if}}{\sigma_{pf}} .$$

In Excel, this slope is calculated by the function **"SLOPE([$\mathbf{r_{if}}$], [$\mathbf{r_{pf}}$])"**. Substituting the expression for beta in (18) gives :

(21) $$1 = \sum_i w_i \beta_{ip}$$

So β_{ip} is the relative marginal contribution of asset *i* to portfolio volatility (or the relative marginal risk contribution) :

(22) $$\beta_{ip} = \frac{\partial \sigma_{pf} / \sigma_{pf}}{\partial w_i}$$

and $w_i\beta_{ip}$ is the asset's relative component contribution to portfolio volatility. So given the assets' betas, the decomposition of portfolio volatility is a piece of cake. When $w_i\beta_{ip}$ is comparatively large, this identifies a "hot spot" in the portfolio, or a pocket of risk concentration, indicating that asset's *i* contribution to portfolio risk is large. Hence, this position is likely to contribute heavily to any loss that may be realized on the portfolio.

Sumarizing : $\{w_i\}$ defines money allocation and $\{w_i \cdot \beta_i\}$ defines risk allocation. To go from money allocation to risk allocation, each investment weight is multiplied with the corresponding beta (note that the average value of beta is unity).

Portfolio optimality : maximize the Sharpe Ratio

From eq.(19) it follows that the expected excess return or **risk premium** of asset *i* is related to the portfolio's risk premium as :

(23) $$\bar{r}_{if} = \alpha_i + \beta_{ip}\bar{r}_{pf}$$

Now consider the mean-variance optimal portfolio, this is the portfolio that maximizes the Sharpe Ratio :

(24) $$\max_{\{w_i\}_{i\in p}} \; SR_p = \frac{\bar{r}_{pf}}{\sigma_{pf}}$$

The first-order conditions for optimality can be shown to imply the following relation between risk premia and betas :

(25) $$\bar{r}_{if} = \beta_{ip}\bar{r}_{pf}$$

In words : for the maximum Sharpe Ratio Portfolio MSRP, the risk premia of all constituents are proportional to their betas. Considering eq.(23), this implies that for all assets included in the MSRP *p**, the alpha α_i equals zero, $\alpha_i = 0 \;\; \forall i \in p^*$. To provide some intuition, note that for each asset comprised in a maximum Sharpe Ratio portfolio the relative marginal contribution to excess return must equal the relative marginal contribution to risk, or :

(26) $$\frac{\bar{r}_{if}}{\bar{r}_{pf}} = \beta_{ip}$$

This can be rephrased as requiring equal ratios of marginal return and risk contributions :[4]

[4] Note that $\bar{r}_{if} / \beta_{ip}$ is the Treynor [1966] ratio of risk-adjusted performance.

(27) $$\frac{\overline{r}_{if}}{\beta_{ip}} = \frac{\overline{r}_{jf}}{\beta_{jp}} = \overline{r}_{pf}$$

If this does not hold, the Sharpe Ratio of the portfolio can be improved by increasing the weight of the assets with higher contributions to return (or lower contributions to risk) and decreasing the weight of assets with lower contributions to return (or higher contributions to risk).

In other words, referring to eq.(23), when an asset's alpha is positive, $\alpha_i > 0$, this asset shows outperformance against the portfolio and the Sharpe Ratio of the portfolio can be increased by increasing the weight of this asset. Conversely, when an asset's alpha is negative, $\alpha_i < 0$, this asset shows underperformance and the portfolio's Sharpe Ratio can be increased by decreasing the weight of this asset. **Summarizing :** in a MSRP, high risk contribution should be matched with high return contribution. A more than proportionate return-to-risk contribution indicates a positive "alpha".

Two additional comments are in order. Firstly, you may recognize in eq.(25) the infamous "Security Market Line" or the Capital Asset Pricing Model **CAPM**. However, the results above apply to **any** maximum Sharpe Ratio portfolio, whereas the CAPM applies to the equally infamous market portfolio (the overall market cap weighted portfolio containing all assets) under the heroic equilibrium assumption that this portfolio is mean-variance efficient. Hence, the results presented above are completely general.

Secondly, using the second definition of beta in eq.(20) allows us to rewrite (25) as $\overline{r}_{if} / \sigma_{if} = \rho_{ip}\overline{r}_{pf} / \sigma_{pf}$. Using the definition of the Sharpe Ratio, this boils down to :

(28) $$SR_i = \rho_{ip} SR_p$$

In words : for any maximum Sharpe Ratio portfolio, any constituent's the stand-alone Sharpe Ratio equals the product of (i) its correlation with this portfolio and (ii) the Sharpe Ratio of the portfolio. When an asset's Sharpe Ratio is larger (smaller), this implies that the asset's alpha is positive (negative). This also applies to assets not comprised in the portfolio. When $\alpha_i > 0$, or equivalently $SR_i > \rho_{ip} SR_p$, the Sharpe Ratio of the portfolio is increased by adding that asset to the portfolio (and *vice versa*).

Reverse optimization : implied risk premia

In conventional mean-variance portfolio optimization, the asset's risk premia and their covariance matrix are used to calculate the weights of the maximum Sharpe Ratio portfolio. In reverse portfolio optimization, it is assumed that the portfolio at hand actually **is** the maximum Sharpe Ratio portfolio. Together with the covariance matrix of excess returns this allows us to derive the "imputed" risk premia (see Sharpe [1974]). Using these implied risk premia together with the asset's standard deviations, we can then compute the implied Sharpe Ratios. Hence, given a particular portfolio, these implied risk premia (or implied Sharpe Ratios) would make this portfolio the maximum Sharpe Ratio portfolio.

How do we derive these implied risk premia ? Accepting the portfolio risk premium as is, we simply use the first-order condition for the MSRP in eq.(25) together with the asset betas to compute the implied risk premium $r_{if}*$ as the product of the beta and the portfolio risk premium :

(29) $$r_{if}^* = \beta_{ip}\overline{r}_{pf}$$

The implied Sharpe Ratio readily follows as $SR_i^* = r_{if}^* / \sigma_{if}$.

Deriving implied risk premia is relevant when there is uncertainty about ex-ante risk premia. It is well-known that composition of the MSRP is very sensitive to the input risk premia; slight differences in these inputs can result in very different (and sometimes “unrealistic” or extreme and hence unacceptable) portfolios. At the same time, estimating ex-ante risk premia is a very difficult task. Reverse optimization can help since the assets’ implied risk premia serve as a sensible starting point. Depending on the confidence placed in one’s ex-ante views, one can next adjust the implied risk premia accordingly. After this two-stage process, the resulting portfolio is closer to the original portfolio and less extreme. This two-stage portfolio optimization process is proposed by Black & Litterman [1992].

12. References

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