Optimal portfolio model under compound jump processes

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Abstract

Purpose – The purpose of this paper is to research the optimal portfolio proportion for the optimal investment model and the optimal consumption investment strategies for the optimal consumption investment model under compound-jump processes.

Design/methodology/approach – Traditionally, the price of risky security or asset is often modeled as geometric Brownian motion. However, the analysis of stock price evolution reveals sudden and rare breaks logically accounted for by exogenous events on information. It is natural to model such behavior by means of a point process, or, more simply, by a Poisson process, which has jumps of constant size occurring at rare and unpredictable intervals. Assume that the price of risky security stock is modeled by a compound-jump process, the renew process theory is chosen to solve the optimal investment model, the HJB equation is chosen for the optimal consumption investment model.

Findings – Derive the analytical optimal portfolio proportion for the reduction model of optimal investment. The optimal consumption investment strategies are given by some equations for the optimal consumption investment model.

Research limitations/implications – Accessibility and availability of data are the main limitations which model will be applied.

Practical implications – The results obtained in this paper could be used as a guide to actual portfolio management.

Originality/value – The new approach for the optimal portfolio model under compound-jump processes. The paper is aimed at actual portfolio managers.

Keywords Cybernetics, Modelling, Portfolio investment, Finance

Paper type Research paper

1. Introduction

Traditionally, the price of risky security or asset is often modeled as geometric Brownian motion (Xu, 2007; Zhang and Yue, 2006; Kong and Ni, 2007). However, the analysis of stock price evolution reveals sudden and rare breaks logically accounted for by exogenous events on information. (To this end, relevant but more general discussions on sudden breaks and abruptions in evolutions can be found in Lin (1998, 2001) and OuYang *et al.* (2001a, b).) From a probabilistic point of view, it is natural to model such behavior by means of a point process, or, more simply, by a Poisson process, which has jumps of constant size occurring at rare and unpredictable intervals. Some researchers have used general discontinuous process (e.g. jump-diffusion process or Levy process) to

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model the asset returns (Emmer *et al.*, 2001; Yan *et al.*, 2000; Jeanblace-picque and Pontier, 1990; Guo and Xu, 2004; Zhang, 1997). For instance, Emmer *et al.* (2001) considers optimal portfolios with bounded capital at risk. In his model, the random fluctuations of stock prices are generated by both a Brownian discontinuous process (e.g. jump-diffusion or Levy process) to model the asset returns (see motion and a compound-jump process. Yan *et al.* (2000) research growth optimal portfolio in a market driven by a jump-diffusion-like or a Levy process. Jeanblace-picque and Pontier (1990) studies optimal portfolio for a small investor in a market model with discontinuous prices. Guo and Xu (2004) study portfolio selection problem in which the prices of stocks follow jump-diffusion process is studied. The objective is to maximize the expected terminal return and minimize the variance of the terminal wealth.

In this paper, we consider a financial market in which there are one riskless asset bond and one risky security stock whose price is modeled by a compound-jump process. We formulate two models: Model 1 is the optimal investment model and Model 2 is the optimal consumption investment model. The criterion for the optimal investment model is to maximizing the long-run growth rate of his investment. The object for the optimal consumption investment model is to maximizing the expected utility from both consumption and terminal wealth. Our aim is to derive the optimal portfolio proportion for Model 1, and the optimal consumption investment strategies for Model 2 under the compound-jump processes. Furthermore, we aim at the explicitly optimal consumption investment strategies for the case of the constant relative risk aversion (CRRA). It should be pointed out that there are some differences between the existing literature and this paper. The main differences lie in:

- This paper considers the investors' investment consumption problem under the compound-jump processes, while Zhang (1997) focuses on the problem of option pricing.
- In our model, we consider two problems: maximizing the long-run growth rate of his investment and maximizing the expected utility from both consumption and terminal wealth, but the objective in Guo and Xu (2004) is to maximize the expected terminal return and minimize the variance of the terminal wealth, the purpose of Yan *et al.* (2000) is to work out the growth optimal portfolio in a market with asset returns being a jump-diffusion-like process or a levy process, and Emmer *et al.* (2001) considers optimal portfolios with bounded capital at risk.
- Though Jeanblace-picque and Pontier (1990) also studies optimal portfolio with the goal of maximizing utility from consumption and terminal wealth, they suppose the prices of risky asset are modeled by a stochastic differential equation with a jump-process component.

But the price of risky asset in this paper is modeled by compound-jump process.

The paper is organized as follows. In Sections 2.1 and 2.2, we formulate the optimal investment model and solve the model. Section 2.3 gives some numerical examples. Sections 3.1 and 3.2 formulate the optimal investment consumption problem under the compound-jump processes and solve the problem. We use the generalized ITO formula to derive the HJB equation based on the compound-jump processes. Section 3.3 studies the CRRA case in order to obtain the explicit optimal consumption investment strategies. Section 3.4 presents some simulation results.

2. The optimal investment model

2.1 The formulation of optimal investment model

We suppose the portfolio involves two assets: the bank account, a stock or stock index. It is without any loss of generality since it is just a simple matter of algebra to generalize our results to multi-stocks models. The price of bond at time t, S_t^0 satisfies $\mathrm{d}S_t^0/S_t^0 = r(t)dt$. The price of stock at time t, S_t satisfies:

$$\mathrm{dS_t/S_{t-}} = b(t)dt + \sigma(t)dW_t + \sum_{i=1}^{n}[\beta_i dN_i(t) - \beta_i\lambda_i dt],$$

where $r(t)$ is the riskless interest rate, $\mu(t)$ is the drift rate, $\sigma(t)$ is the volatility, they are all determined functions. W_t is the standard Brownian motion with the right continuous Brownian filtration F_t, and $\mathrm{W}_0 = 0$. The process N_i is a Poisson process with intensity λ_i, β_i is the jump height of the process N_i. Denote the wealth value at time t by V(t), V(0) $=$ v > 0. Let $\pi(t)$ be the proportion of V(t) in the stock. It follows that:

$$\frac{dV_t}{V_{t-}} = [(1-\pi(t))r(t) + \pi(t)b(t)]dt + \pi(t)\sigma(t)dW_t + \pi(t)\sum_{i=1}^{n}[\beta_i dN_i(t) - \beta_i\lambda_i dt]. \quad (1)$$

Definition 1. We call $\pi(t)$ an admissible strategy if $\pi(t)$ is non-negative, progressively measurable with respect to $\{F_t\}_{t\geq 0}$ and satisfies $\int_0^T \pi(t)dt < \infty$ a.s., where F_t represents the σ-algebra generated by $(W_s, N(s))$: s $\leq$ t. Denote all the admissible strategies set by $\prod$.

As we all know, the long-run growth rate of investment is:

$$J(\pi(t)) = \frac{\lim\limits_{T\to\infty} E[\ln V(T)]}{T}. \quad (2)$$

Problem 1. The investment criterion is to choose an admissible policy to maximize the long-run growth rate of his investment (this is the Kelly criterion). That is:

$$J(\pi^*(t)) = \max_{\pi(t)\in\Pi} J(\pi(t)) = \frac{\max\limits_{\pi(t)\in\Pi}\lim\limits_{T\to\infty} E[\ln V(T)]}{T}. \quad (3)$$

2.2 Solving problem 1

Lemma 1. Suppose that v_{ij} is the jth jump time, N_T is the total jump times of the process $N_i(t)$ on [0,T], then:

$$E\left[\sum_{j=1}^{N_T}\ln(1+\pi(v_{ij})\beta_i)\right] = \lambda_i\int_0^T \ln(1+\pi(t)\beta_i)dt. \quad (4)$$

Proof. Applying the Renewal theorem and Theorem 2.3.1 of Ross (1996), we can prove Lemma 1 simply. □

Theorem 1. The optimal strategy $\pi^*(t)$ must satisfy:

$$\sum_{i=1}^{n} \frac{\lambda_i \beta_i}{1+\pi(t)\beta_i} - \pi(t)\sigma^2 - \sum_{i=1}^{n} \lambda_i \beta_i + b(t) - r(t) = 0. \tag{5}$$

Proof. By equation (1) and ITO formula, one has:

$$E\ln(V_T) = \ln(V_0) + E\left\{ \int_0^T \left[\frac{(1-\pi(t))r(t) + \pi(t)b(t) - \pi(t)\sum_{i=1}^{n}\beta_i\lambda_i - \pi^2(t)\sigma^2(t)}{2} \right] dt + \sum_{i=1}^{n} \int_0^T \ln(1+\pi(t)\beta_i)dN_i(t) \right\}. \tag{6}$$

By Lemma 1, we obtain:

$$E\left[\sum_{i=1}^{n}\int_0^T \ln(1+\pi(t)\beta_i)dN_i(t)\right] = \sum_{i=1}^{n} E\left[\int_0^T \ln(1+\pi(t)\beta_i)dN_i(t)\right] = \sum_{i=1}^{n}\lambda_i \int_0^T \ln(1+\pi(t)\beta_i)dt. \tag{7}$$

Combining equation (6) with equation (7), it follows that:

$$E\ln(V_T) = \ln(V_0) + E\int_0^T \left\{ \left[\frac{(1-\pi(t))r(t) + \pi(t)b(t) - \pi(t)\sum_{i=1}^{n}\beta_i\lambda_i - \pi^2(t)\sigma^2(t)}{2} \right] + \sum_{i=1}^{n}\lambda_i \ln(1+\pi(t)\beta_i) \right\} dt. \tag{8}$$

Assume $F(t, \pi(t)) = \dfrac{(1-\pi(t))r(t) + \pi(t)b(t) - \pi(t)\sum_{i=1}^{n}\beta_i\lambda_i - \pi^2(t)\sigma^2(t)}{2} + \sum_{i=1}^{n}\lambda_i \ln(1+\pi(t)\beta_i).$

By virtue of equation (2) and L' Hospital Law, one has:

$$J(\pi(t)) = \frac{\lim_{T\to\infty} E[\ln V(T)]}{T} = \frac{\lim_{T\to\infty} \int_0^T F(t, \pi(t))dt}{T} = \lim_{t\to\infty} F(t, \pi(t)). \tag{9}$$

Using equation (9), to max equation (2), we have $(\partial F(t, \pi(t)))/\partial\pi = 0$. After rearranging, we get equation (5). □

2.3 The analytical solution for the reduction model

We consider n = 1. Suppose the stock price satisfies:

$$\frac{dS_t}{S_t} = bdt + \sigma dW_t + [\beta dN(t) - \beta\lambda dt].$$

By Theorem 1, the optimal proportion:

$$\pi^*(t) = \frac{\{[(-\sigma^2 + \beta(-r + b - \beta\lambda)] + \sqrt{\Delta}\}}{(2\sigma^2\beta)},$$

where:

$$\Delta = [-\sigma^2 + \beta(-r + b - \beta\lambda)]^2 + 4\sigma^2\beta(b - r) \geq 0.$$

It suggests that the optimal proportion is still fixed proportion. The investor has to dynamically adjust the fraction of the wealth allocated to the stock in order to maximize the investment long run growth rate. Moreover, the Merton Line is still a line.

2.4 The simulation results

In this section, we set a numerical application in order to analyse the dynamic behaviour of the optimal portfolio strategy derived in Section 2.3. Especially, we focus on the influence of jump height and intensity on the optimal strategy. The set of parameters is as follows: r = 0.05, $\sigma = 0.2$, b = 0.07. We set that β takes the value of -0.1, 0.1, respectively. λ = 0.3, 0.5, 1, respectively. (That is, the price of the stock jumps one time every three years, one time every two years, one time every year, respectively.) The simulation results are in Figures 1 and 2.

It is very clear from Figure 1 that, in spite of the jump direction of the stock, the optimal portfolio proportions increase with the rise of the Poisson intensity λ. For example, when $\lambda = 0.7$, if the stock price jumps down wards, $\beta = -0.1$,

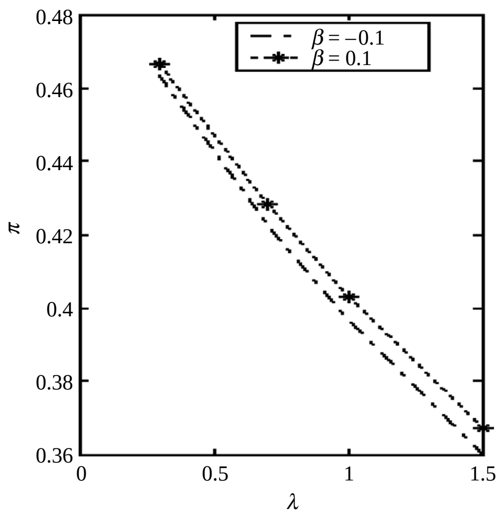

Figure 1.
π^* depending on λ

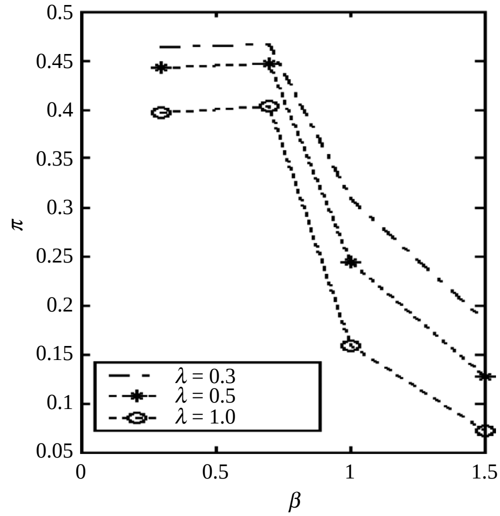

Figure 2. π^* depending on β

then $\pi^* = 0.4228$; whereas if the stock price jumps upwards, $\beta = 0.1$, then $\pi^* = 0.4282 > 0.4228$. This is accordance with our intuition: because that the price of stock is falling frequently, the investor sure sells out the stock and holds the riskless bond. Figure 2 shows that the optimal portfolio proportions increase with the rise of the jump height β. For example, when $\lambda = 0.3$, if $\beta = 0.1$, $\pi^* = 0.4666$, if $\beta = 0.3$, then $\pi^* = 0.3091 < 0.4666$; when $\lambda = 0.5$, $\beta = 0.1$, $\pi^* = 0.4466 < 0.4666$, that is, when Poisson intensity λ increases from 0.3 to 0.5, the optimal proportion the investor holds in the stock decreases from 0.4666 to 0.4466.

3. The optimal investment consumption model

3.1 The formulation of optimal investment consumption model

We also suppose the portfolio involves two assets: the bank account, a stock or stock index. The asset-price equations are same to equations (1) and (2). Let $\pi(t)$ be the proportion of V(t) in the stock, c(t) be the rate at which the investor withdraws funds for consumption. It follows from equations (1) and (2) that:

$$\begin{aligned} dV_t = {} & \{V(t)[(1-\pi(t))r(t)+\pi(t)b(t)] - c(t)\}dt + \pi(t)V(t)\sigma(t)dW_t \\ & + \pi(t)V(t)\sum_{i=1}^{n}[\beta_i dN_i(t) - \beta_i\lambda_i dt]. \end{aligned} \tag{10}$$

Definition 2. We call $(c(t), \pi(t))$ an admissible consumption-investment strategy if $\pi(t)$ is progressively measurable with respect to $\{F_t\}_{t\geq 0}$ and satisfies $\int_0^T \pi(t)dt < \infty$ a.s., the process c(t) is non-negative, progressively measurable with respect to $\{F_t\}_{t\geq 0}$ and satisfies $\int_0^T c(t)dt < \infty$ a.s. Let $\mathbf{H}$ denotes the set of all admissible consumption investment policies.

To formulate the meaningful optimization problem, we shall need the usual assumption for the utility function (Karatzas, 1989). The expected utility from both consumption and terminal wealth is:

$$J(c, \pi; x, v) := E\left[\int_s^T U(t, c(t))dt + L(T, V(T))\right]. \tag{11}$$

Problem 2. The investment criterion is to choose an admissible policy $(c(t), \pi(t)) \in H$ to maximize the expected utility from both consumption and terminal wealth, i.e.:

$$J(\bar{c}, \bar{\pi}; x, v) = \max_{(c,\pi)\in H} J(c, \pi; x, v) = \max_{(c,\pi)\in H} E\left[\int_t^T U(t, c(t))dt + L(T, V(T))\right] \tag{12}$$

3.2 Solving problem 2

Theorem 2. Assume $W \in \{W(t, v) : (\partial W/\partial t), (\partial W/\partial v), (\partial^2 W/\partial v^2),$ exist and continuous, there are some constants c, k such that $|W(t, v)| < c(1 + |v|^k)\}$ satisfies the following HJB equation:

$$\max_{(c,\pi)\in H} \{A^v W(t, v) + U(t, c(t)\} = 0, \quad W(T, v) = L(T, v). \tag{13}$$

If $(\bar{c}, \bar{\pi}) \in H$ and:

$$(\bar{c}, \bar{\pi}) \in \arg \max_{(c,\pi)\in H} \{A^v W(t, v) + U(t, c(t)\},$$

then:

$$W(t, \bar{v}) = \max_{(c,\pi)\in H} J(c, \pi; x, v),$$

where $\bar{v}(\cdot)$ is the corresponding trajectory of $c = \bar{c}$, $\pi = \bar{\pi}$ in equation (10), $(\bar{c}, \bar{\pi})$ is the optimal consumption-investment strategy of problem (2).

Proof. The proof is very similar to Theorem 1 of Guo and Xu (2004). □

Theorem 3. The optimal consumption-investment strategy of problem (2) must satisfy:

$$\frac{\partial U(t, c)}{\partial c} = \frac{\partial W(t, v)}{\partial v}, \tag{14}$$

$$\frac{\partial W(t, v)}{\partial v} v\left[b(t) - r(t) - \sum_{i=1}^{n} \beta_i \lambda_i\right] + \frac{\partial W^2(t, v)}{\partial v^2} \pi(t)\sigma^2(t)v^2 + \sum_{i=1}^{n} \lambda_i \left[\frac{\partial W(t, v(1 + \pi\beta_i))}{\partial v} v\beta_i\right] = 0. \tag{15}$$

Proof. By generalized ITO formula (Jeanblace-picque and Pontier, 1990), equation (15) and the first-order conditions, we can easily get Theorem 3. □

3.3 The explicitly optimal solution for the CRRA case

The case of CRRA is to suppose that the utility function has the following form:

$$U(t,c)=\frac{Me^{-\gamma t}c^{1-R}}{(1-R)},\quad L(T,V(T))=\frac{Ne^{-\gamma T}V(T)^{1-R}}{(1-R)},$$

where M, N, R, γ are all constants, $0 < R < 1$.

Theorem 4. Under the CRRA case, the optimal consumption-investment strategies of problem (2) are given by equations (17) and (18), the optimal expected utility by equation (16).

Proof. As a solution to equation (13), we try:

$$\Phi(t,v)=f(t)v^{1-R},\quad f(T)=\frac{e^{-\gamma T}N}{(1-R)}. \tag{16}$$

By some simple calculation, we have:

$$\bar{c}(t)=\left[\frac{f(t)(1-R)e^{\gamma t}}{M}\right]^{-1/R}V(t), \tag{17}$$

$$b(t)-r(t)-\sum_{i=1}^{n}\beta_i\lambda_i-\bar{\pi}(t)\sigma^2(t)+\sum_{i=1}^{n}\lambda_i(1+\bar{\pi}\beta_i)^{1-R}\beta_i=0 \tag{18}$$

where:

$$f(t)=\exp\left(\int_t^T f_2(s)ds\right)\tilde{f}(t)>0, f_2(t)$$

$$=(1-R)\left[\frac{\left[r(t)+\bar{\pi}(t)(b(t)-r(t))-\bar{\pi}(t)\sum_{i=1}^{n}\beta_i\lambda_i-R\bar{\pi}^2(t)\sigma^2(t)\right]}{2}\right]$$

□

3.4 The simulation results for the case of CRRA

In order to further recognize the optimal consumption-investment strategies under compound-jump processes obtained in Section 3.3, we consider the case of n = 1. Assume that the stock price satisfies:

$$\frac{dS_t}{S_t}=bdt+\sigma dW_t+[\beta dN(t)-\beta\lambda dt].$$

By equation (18), the optimal proportion is given by:

$$b(t)-r(t)-\beta\lambda-\bar{\pi}(t)\sigma^2(t)+\lambda(1+\bar{\pi}\beta)^{1-R}\beta=0.$$

If we assume M = N = R = γ = 0.5, T = 2, r = 0.05, σ = 0.2, b = 0.07, then by equation (17), we get the optimal consumption rate is:

$$\bar{c}(t)=\frac{\exp(-0.0844-0.9578t)V(t)}{[-0.1413+1.0441\exp(-0.0844-0.9578t)]^2}.$$

Next, we set the basic market parameters are r = 0.05, $\sigma = 0.2$, b = 0.07. Figures 3-5 show the simulation results for π^*,β, γ and R (using Matlab 7.0).

From Figure 3, in spite of the jump direction of the stock, the optimal portfolio proportions increase with the rise of the Poisson intensity λ. Furthermore, the optimal portfolio proportions with downward jump are very close to the ones with upward jump. For example, when $\lambda = 0.7$, if the stock price jumps down wards, $\beta = -0.1$, $\pi^* = 0.54868$, whereas if the stock price jumps upwards, $\beta = 0.1$, then $\pi^* = 0.547246 < 0.54868$. But this is not in accordance with our economic intuition:

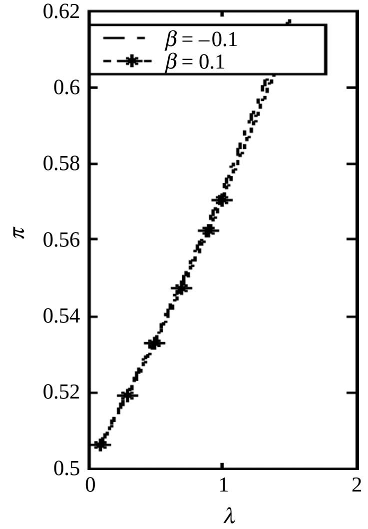

Figure 3.
π^* depending on λ

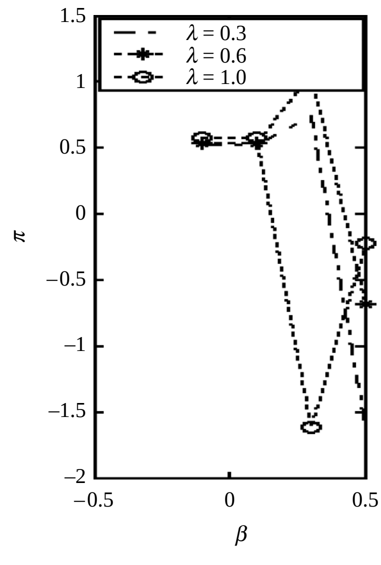

Figure 4.
π^* depending on β

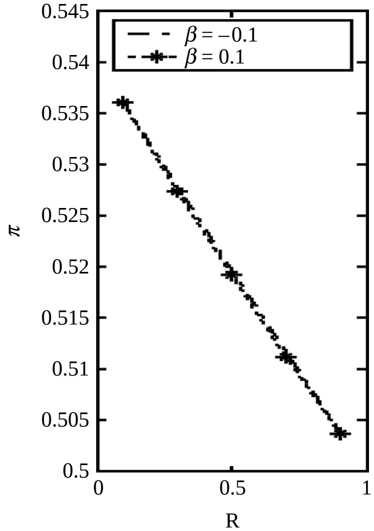

Figure 5.
π^* depending on R

since if the price of stock is falling, generally, the investors sell out the stock and hold the riskless bond, the proportion allocated in the stock will be reduced. Figure 4 shows that the optimal portfolio proportions react rulelessly to the jump height β. It can be seen from Figure 5 that, in spite of the jump direction of the stock, the optimal portfolio proportions decrease with the increase of the risk aversion parameter R. For example, when $\beta = -0.1$, $\lambda = 0.3$, if $R = 0.3$, then $\pi^* = 0.52794$; if $R = 0.9$, then $\pi^* = 0.50386 < 0.52794$. The bigger the risk aversion parameter R, the less the optimal portfolio proportions. This result is consistent with economic intuition. Note that even such a big value of the risk aversion parameter R as $R = 0.9$ does not cause a significant change in the optimal proportions.

4. Concluding remarks

This paper considers a financial market in which there is a single riskless bond and a risky stock modeled by compound-jump processes. We formulate two models: Model 1 is the optimal investment model and Model 2 is the optimal investment consumption model. The criterion for the optimal investment model is to maximizing the long-run growth rate of his investment. The object for the optimal investment consumption model is to maximizing the expected utility from both consumption and terminal wealth. We solve the Model 1 by the renewal process theory and obtain the corresponding HJB equation for Model 2 based on the compound-jump processes. The results obtained in this paper could be used as a guide to actual portfolio management. If there is friction in the market, how the optimal strategy will be changed is a subject for future research.

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