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The Gerber–Shiu discounted penalty function in the classical risk model with impulsive dividend policy[☆]

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ABSTRACT

In this paper, we study the Gerber–Shiu discounted penalty function in the classical risk model with impulsive dividends. When the surplus process hits a barrier b, the dividend will be paid and the surplus is reduced to a level a. An integro-differential equation for the Gerber–Shiu discounted penalty function is derived by analyzing the evolution of the surplus process and it is solved by Dickson–Hipp operator method. For this process, we also investigate the Laplace transform of the time of ruin, the distribution of the surplus immediately before ruin and the deficit at ruin. These quantities for the special case where the claim size is exponentially distributed are obtained explicitly. Moreover, the distribution of the number of dividends is derived.

1. Introduction

To give a rigorous mathematical formulation of the problem, we start with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$. In the absence of dividends, the surplus process X_t of an insurance company is given by

$$X(t) = \begin{cases} x + ct - \sum_{i=1}^{N(t)} Y_i, & \text{if } x < b \\ a, & \text{if } x = b \end{cases} \tag{1.1}$$

where $x \geq 0$ is the initial surplus, $c > 0$ is the premium rate, $\{N(t)\}$ is a Poisson process with intensity $\lambda > 0$, $\{Y_i\}$ is an i.i.d. sequence of strictly positive random variables with distribution function Q and probability density function q. The claim sizes $\{Y_i\}$ and the claim arrival process $\{N(t)\}$ are assumed to be independent. We also assume that $\mathrm{E}[Y_i] = \mu < \infty$. Now, we enrich the model and consider the dividend payments. When the fixed cost is taken into account, the strategy becomes impulsive. The controlled surplus process evolves as follows. Whenever the surplus reaches a barrier b it is reduced to a level a through a dividend payment. If we want to indicate that the initial surplus is x, we will write P_x and E_x for the probability measure and the expectation, respectively. The time of ruin is defined by

$$\tau = \inf\{t \geq 0 : X_t \leq 0\}. \tag{1.2}$$

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This model has been derived by Bai and Guo (2010) as the surplus process corresponding to the optimal dividend payoff (see Bai and Guo, 2010 for details).

We now introduce the Gerber–Shiu discounted penalty function

$$m(x) = \mathrm{E}_x[e^{-\delta\tau}\omega(X(\tau-), |X(\tau)|)\mathrm{I}(\tau < \infty)] \tag{1.3}$$

where $\delta > 0$ is interpreted as the force of interest, ω is a nonnegative function on $\mathbb{R}_+^2$, and $\mathrm{I}(A)$ is the indicator function of an event A. A number of particular cases of the discounted penalty function lead to important quantities of interest in risk theory. For instance, setting $\omega \equiv 1$, we obtain the Laplace transform of the time of ruin. When $\delta = 0$ and $\omega \equiv 1$, the discounted penalty function reduces to the probability of ultimate ruin.

Since Gerber and Shiu (1998), the Gerber–Shiu function was studied by many authors. A large number of papers extended the study to more practical situations by introducing certain economic factors such as dividends. The study of the Gerber–Shiu function of the classical risk model within the framework of the so-called threshold or the barrier dividend strategy has received considerable attention in the actuarial literature. For example, Lin et al. (2003) analyzed the Gerber–Shiu discounted penalty function of the classical compound Poisson risk model with a constant dividend barrier. When the distribution of the claim sizes is exponential or a mixture of two exponentials, explicit solution was obtained. Lin and Pavlova (2006) investigated the expected penalty function with a threshold dividend strategy. Lin and Sendova (2008) examined the expected discounted penalty function in a multi-threshold compound Poisson risk model. Cai and Dickson (2002) considered the corresponding problem of the classical surplus process modified by the inclusion of interest. Yuen et al. (2007) studied the problem for the risk process with interest and a constant dividend barrier.

However, in these papers, it was assumed that there were no transaction costs when dividends were paid out which are different from ours. Firstly, because of the fixed transaction cost, the optimal strategy obtained by Bai and Guo (2010) is impulsive. On the other hand, because of the impulsive nature of the dividend strategy, the transaction cost can be combined with the dividend strategy by deducting this cost from the dividend. Secondly, the evolution of the surplus process differs from those in the references. In the case that the two thresholds tend to infinity, the surplus process is the same as the classical risk model. In the case where the lower threshold becomes 0, the evolution of the surplus process is the same as the one under barrier strategy. None of these work examines the Gerber–Shiu function under impulsive strategy. This motivates us to study the problem in this paper.

The rest of the paper is organized as follows. In Section 2, we obtain an integro-differential equation for $m(x)$. In Section 3, it is shown that the Gerber–Shiu discounted penalty function under the impulsive strategy is a linear combination of a solution to a homogeneous integro-differential equation and the Gerber–Shiu discounted penalty function without dividend. In Section 4, the Laplace transform of the time of ruin, the distribution of the surplus immediately before ruin and the deficit at ruin are obtained. The distribution of the number of dividend payments is presented in Section 5.

2. An integro-differential equation

In this section, we derive an integro-differential equation for $m(x)$.

Proposition 2.1. *The Gerber–Shiu function m satisfies the following ODE*

$$cm'(x) - (\lambda+\delta)m(x) + \lambda\int_0^x m(x-y)\mathrm{d}Q(y) + \lambda\int_x^\infty \omega(x, y-x)\mathrm{d}Q(y) = 0, \quad 0 \le x \le b, \tag{2.1}$$

with boundary condition

$$m(b) = m(a). \tag{2.2}$$

Proof. Let $t_0 = \frac{b-x}{c}$ and $t_i = t_0 + i\frac{b-a}{c}$, $i \ge 1$. Without claims, the surplus process reaches barrier b at each t_i. For simplicity of notation, we denote

$$\xi = e^{-\delta\tau}\omega(X(\tau-), |X(\tau)|)\mathrm{I}(\tau < \infty).$$

Let τ_1 be the arrival time of the first claim. Then, $m(x) = \sum_{i=1}^\infty J_i$, where

$$J_0 = \mathrm{E}_x(\xi\mathrm{I}(\tau_1 < t_0)) \quad \text{and} \quad J_i = \mathrm{E}_x(\xi\mathrm{I}(t_{i-1} \le \tau_1 < t_i)), \quad \forall\, i \ge 1.$$

According to the cases of ruin or not, we can write J_0 as

$$J_0 = \int_0^{t_0}\int e^{-\delta t} m(x+ct-y)\mathrm{I}(x+ct-y \ge 0)Q(dy)\lambda e^{-\lambda t}dt$$
$$+ \int_0^{t_0}\int e^{-\delta t}\omega(x+ct, y-x-ct)\mathrm{I}(x+ct-y < 0)Q(dy)\lambda e^{-\lambda t}dt.$$

Letting $s = x + ct$, we get

$$J_0 = \int_x^b \int_0^\infty \frac{\lambda}{c} I(s-x, y) Q(dy) ds,$$

where

$$I(s, y) = e^{-\frac{\lambda+\delta}{c}s}\left(m(s-y)1_{y<s} + \omega(s, y-s)1_{y>s}\right).$$

Similarly, for $i \geq 0$, we can write

$$J_i = \int_a^b \int_0^\infty \frac{\lambda}{c} I(s-a-ct_{i-1}, y) Q(dy) ds.$$

Taking summation over $i \geq 1$ and $i = 0$, we get

$$m(x) = \int_0^\infty \int_a^b \left(\frac{\lambda}{c} 1_{x<s<b} + \kappa\right) I(s-x, y) ds Q(dy), \tag{2.3}$$

where $\kappa = \lambda c^{-1}\left(1 - e^{-(\lambda+\delta)(b-a)/c}\right)^{-1} e^{-\frac{\lambda+\delta}{c}(b-a)}$.

Taking derivative on both sides of (2.3), we obtain the expression for $m'(x)$. Eq. (2.1) is then verified by substituting (2.3) into this expression. The boundary condition (2.2) is trivial since the surplus process is reduced to level a once it reaches the barrier b. ■

Remark 2.1. Lin et al. (2003) studied a similar model with continuous dividend strategy rather than an impulsive one. The difference in the dividend strategy leads to different boundary conditions. Instead of (2.2), there is only one boundary in their equation and the solution satisfies Neumann boundary condition.

3. Solution for $m(x)$

Let $u(x)$ be a specific solution to (2.1). Let $v(x)$ be a solution to the homogeneous version of (2.1), namely,

$$cv'(x) - (\lambda + \delta)v(x) + \lambda \int_0^x v(x-y) \mathrm{d}Q(y) = 0. \tag{3.1}$$

Then, considering the boundary condition (2.2), we can write $m(x)$ as follows:

$$m(x) = u(x) + \frac{u(a) - u(b)}{v(b) - v(a)} v(x), \quad 0 \leq x \leq b. \tag{3.2}$$

Now we derive the expressions of u and v. The Dickson–Hipp operator is used to facilitate solving integro-differential equation (2.1). As in Dickson and Hipp (2001), we define an operator $\mathcal{T}_s$ of a real-valued function $f(x)$.

Definition 3.1. For any integrable function f defined on $[0, \infty)$ and $s \geq 0$, the Dickson–Hipp transform of f is given by

$$\mathcal{T}_s f(x) = e^{sx} \int_x^\infty e^{-sy} f(y) dy, \quad x \geq 0.$$

The following identities will be useful in our calculations. We collect them here for the convenience of the reader. The proof of (i) can be found in Section 3 of Dickson and Hipp (2001), and that of (ii) can be found in Lemma A.2 in Cai et al. (2009). The identity (iii) can be checked by simple calculation.

Proposition 3.1. (i) *For any* $s_1, s_2 > 0$, *we have* $\mathcal{T}_{s_1}\mathcal{T}_{s_2} f(x) = \frac{\mathcal{T}_{s_2} f(x) - \mathcal{T}_{s_1} f(x)}{s_1 - s_2}$.

(ii) $\mathcal{T}_s\{f * g\}(x) = \tilde{g}(s)\mathcal{T}_s f(x) + \mathcal{T}_s g * f(x)$, *where the notation* $f * g$ *stands for the convolution of* f *and* g, $\tilde{g}(x)$ *is the Laplace transform of* g.

(iii) $(s\mathcal{I} - \mathcal{D})\mathcal{T}_s f(x) = f(x)$, *where the notation* $\mathcal{I}$ *stands for the identify operator and* $\mathcal{D}$ *for the differentiation operator.*

Theorem 3.1. *The solution* $m(x)$ *to* (2.1) *is given by* (3.2) *with*

$$u = \mathcal{L}^{-1}\left\{\frac{\lambda \mathcal{L}\mathcal{T}_\rho \zeta}{c - \lambda \mathcal{L}\mathcal{T}_\rho q}\right\} \quad \text{and} \quad v(x) = \mathcal{L}^{-1}\left\{\frac{cs}{\lambda + \delta - cs - \lambda \mathcal{L}Q(s)}\right\}, \tag{3.3}$$

where $\zeta(x) = \int_x^\infty \omega(x, y-x) \mathrm{d}Q(y)$ *and* $\mathcal{L}^{-1}$ *denotes the inverse Laplace transform.*

Proof. We rewrite Eq. (2.1) in terms of operators:

$$\left(\frac{\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)u(x)=\frac{\lambda}{c}u*q(x)+\frac{\lambda}{c}\zeta(x). \tag{3.4}$$

Applying Proposition 3.1(iii), we have

$$u(x)=\frac{\lambda}{c}\mathcal{T}_{\frac{\lambda+\delta}{c}}\{u*q+\zeta\}(x). \tag{3.5}$$

Note that, by (ii) of Proposition 3.1,

$$\mathcal{T}_\rho\{u*q+\zeta\}(x)=\widetilde{q}(\rho)\mathcal{T}_\rho u(x)+\mathcal{T}_\rho q*u(x)+\mathcal{T}_\rho\zeta(x), \tag{3.6}$$

where the constant ρ is the solution to the following Lundberg fundamental equation

$$\frac{\lambda}{c}\widetilde{q}(\rho)=\frac{\lambda+\delta}{c}-\rho. \tag{3.7}$$

Hence, by (3.5) and (i) of Proposition 3.1, we have

$$\begin{aligned}u(x)&=\frac{\lambda}{c}\left[\mathcal{T}_\rho\{u*q+\zeta\}(x)-\left(\frac{\lambda+\delta}{c}-\rho\right)\mathcal{T}_\rho\mathcal{T}_{\frac{\lambda+\delta}{c}}\{u*q+\zeta\}(x)\right]\\&=\frac{\lambda}{c}\left[\widetilde{q}(\rho)\mathcal{T}_\rho u(x)+\mathcal{T}_\rho q*u(x)+\mathcal{T}_\rho\zeta(x)-\left(\frac{\lambda+\delta}{c}-\rho\right)\mathcal{T}_\rho\mathcal{T}_{\frac{\lambda+\delta}{c}}\{u*q+\zeta\}(x)\right].\end{aligned} \tag{3.8}$$

In view of (3.5) and (3.7), we obtain

$$\widetilde{q}(\rho)\mathcal{T}_\rho u(x)-\left(\frac{\lambda+\delta}{c}-\rho\right)\mathcal{T}_\rho\mathcal{T}_{\frac{\lambda+\delta}{c}}\{u*q+\zeta\}(x)=0.$$

Hence, $u(x)$ satisfies the defective renewal equation

$$u(x)=\frac{\lambda}{c}\{\mathcal{T}_\rho q*u(x)+\mathcal{T}_\rho\zeta(x)\}.$$

Taking the Laplace transforms, we have $\widetilde{u}(s)=\frac{\lambda}{c}\widetilde{u}(s)\mathcal{L}\mathcal{T}_\rho q(s)+\frac{\lambda}{c}\mathcal{L}\mathcal{T}_\rho\zeta(s)$, where $\mathcal{L}$ denotes the Laplace transform. Thus,

$$\widetilde{u}(s)=\frac{\lambda\mathcal{L}\mathcal{T}_\rho\zeta(s)}{c-\lambda\mathcal{L}\mathcal{T}_\rho q(s)}. \tag{3.9}$$

u is obtained by taking the inverse Laplace transform.

We now give an explicit solution to the homogeneous equation (3.1). Taking the Laplace transform, we get

$$cs(1+\tilde{v}(s))-(\lambda+\delta)\tilde{v}(s)+\lambda\tilde{v}(s)\tilde{Q}(s)=0.$$

This implies $\tilde{v}(s)=\frac{cs}{\lambda+\delta-cs-\lambda\tilde{Q}(s)}$. Consequently, v is obtained by the inverse Laplace transform. ■

Theorem 3.2. *When the claims are exponentially distributed with mean $1/\beta$, we have for $0\le x\le b$,*

$$m(x)=u(x)+\frac{u(a)-u(b)}{v(b)-v(a)}((\beta+d_+)e^{d_+x}-(\beta+d_-)e^{d_-x}) \tag{3.10}$$

where

$$u(x)=\mathcal{L}^{-1}\left\{\frac{\lambda(\beta+\rho)(\beta+s)}{(s-\rho)[c(\beta+\rho)(\beta+s)-\lambda\beta]}[\widetilde{\zeta}(\rho)-\tilde{\zeta}(s)]\right\}, \tag{3.11}$$

and

$$v(x)=(\beta+d_+)e^{d_+x}-(\beta+d_-)e^{d_-x}, \tag{3.12}$$

with

$$d_\pm=\frac{\lambda+\delta-\beta c\pm\sqrt{(\lambda+\delta-\beta c)^2+4\beta c\delta}}{2c}. \tag{3.13}$$

Proof. When Q is the distribution function of an exponential random variable with mean $1/\beta$, we have

$$\mathcal{L}\mathcal{T}_\rho q(s) = \frac{\mathcal{L}q(\rho) - \mathcal{L}q(s)}{s-\rho} = \frac{\beta}{(\beta+\rho)(\beta+s)}$$

where the first equality resulted from that fact that Laplace transform is a special case of the Dickson–Hipp operator. Similarly,

$$\mathcal{L}\mathcal{T}_\rho \zeta(s) = \frac{\mathcal{L}\zeta(\rho) - \mathcal{L}\zeta(s)}{s-\rho} = \frac{\widetilde{\zeta}(\rho) - \widetilde{\zeta}(s)}{s-\rho}.$$

Substituting the expression for $\mathcal{L}\mathcal{T}_\rho q(s)$ and $\mathcal{L}\mathcal{T}_\rho \zeta(s)$ in (3.9) and rearranging terms, we have (3.11) holds.

Multiplying both sides of (3.1) by $\beta\mathcal{I} + \mathcal{D}$ gives

$$cv''(x) + (\beta c - (\lambda+\delta))v'(x) - \delta\beta v(x) = 0.$$

The solution to this ordinary differential equation is given by

$$v(x) = C_1 e^{d_+ x} + C_2 e^{d_- x}. \tag{3.14}$$

Setting $x = 0$ in (3.1) and using (3.14), we get $\frac{C_1}{\beta+d_+} + \frac{C_2}{\beta+d_-} = 0$. Taking $C_1 = \beta + d_+$, we get (3.12). In view of (3.2), (3.3) and (3.12), we see that (3.10) and (3.11) hold. ■

4. On the time value of ruin τ, the surplus before ruin $X(\tau-)$, and the deficit at ruin $X(\tau)$

In this section, we calculate some actuarial quantities in the case where the claims are exponentially distributed. First, we consider the ruin time distribution.

Proposition 4.1. *The Laplace transform of the ruin time is given by*

$$\mathrm{E}e^{-\delta\tau} = u(x) + \frac{u(a)-u(b)}{v(b)-v(a)}v(x), \quad 0 \le x \le b, \tag{4.1}$$

where v is given by (3.12) and

$$u(x) = \frac{\lambda}{c(\beta+\rho)} e^{-(\beta - \frac{\lambda\beta}{c(\beta+\rho)})x}. \tag{4.2}$$

As a consequence, we have $P(\tau < \infty) = 1$.

Proof. We now recall (1.3) and apply Theorem 3.2 to the case $\omega = 1$. In this setting, $\zeta(x) = e^{-\beta x}$ and hence

$$\widetilde{u}(s) = \frac{\lambda}{c(\beta+\rho)(\beta+s) - \lambda\beta}. \tag{4.3}$$

$u(x)$ is obtained by taking the inverse Laplace transform on both sides of (4.3). $m(x)$ is then obtained from (3.2).

Taking $\delta = 0$ in (3.7) and using the fact that $\tilde{q}(\rho) = (\beta+\rho)^{-1}$, we get $\beta + \rho = \frac{\lambda}{c}$. This implies $P(\tau < \infty) = m(x) = 1$. (4.1) then follows from (3.2). ■

Remark 4.1. With the explicit representation of the solution, it can be proved that Laplace transform of the ruin time in our setting is larger that those of Lin et al. (2003). Hence, the ruin time in our model is stochastically smaller than that of Lin et al. (2003). This confirms with our intuition because our model corresponds to the optimal strategy with transaction cost while for that of Lin et al. (2003) no transaction cost need to be paid.

We next turn our attention to the distribution of the surplus $X(\tau-)$ before ruin.

Proposition 4.2. *For any bounded continuous function ω_1, we have*

$$\begin{aligned}\mathrm{E}_x\omega_1(X(\tau-)) = {} & \left(\frac{\lambda\beta}{c\beta-\lambda} - \frac{\lambda^2}{c(c\beta-\lambda)}e^{(\frac{\lambda}{c}-\beta)x}\right)\tilde{\zeta}(0) - \int_0^x \left(\frac{\lambda\beta}{c\beta-\lambda} - \frac{\lambda^2}{c(c\beta-\lambda)}e^{(\frac{\lambda}{c}-\beta)(x-y)}\right)\zeta(y)\mathrm{d}y \\ & + \frac{u_r(a)-u_r(b)}{\lambda e^{\frac{\lambda-c\beta}{c}a} - \lambda e^{\frac{\lambda-c\beta}{c}b}}(c\beta - \lambda e^{\frac{\lambda-c\beta}{c}x}),\end{aligned} \tag{4.4}$$

where $\zeta(x) = \omega_1(x)e^{-\beta x}$ and

$$u_r(x) = \left(\frac{\lambda\beta}{c\beta-\lambda} - \frac{\lambda^2}{c(c\beta-\lambda)}e^{(\frac{\lambda}{c}-\beta)x}\right)\tilde{\zeta}(0) - \int_0^x \left(\frac{\lambda\beta}{c\beta-\lambda} - \frac{\lambda^2}{c(c\beta-\lambda)}e^{(\frac{\lambda}{c}-\beta)(x-y)}\right)\zeta(y)\mathrm{d}y. \tag{4.5}$$

Proof. Taking $\delta = 0$ and $\omega(x_1, x_2) = \omega_1(x_1)$, the corresponding Gerber–Shiu function $m_r(x)$ is given by (3.10) with v_r being given by (3.14), and

$$\widetilde{u}_r(s) = \frac{\lambda(\beta+s)}{s[c(\beta+s)-\lambda]}[\widetilde{\zeta}(0) - \widetilde{\zeta}(s)]. \tag{4.6}$$

(4.5) follows by taking the inverse Laplace transform on (4.6). An easy calculation of $d_\pm$ implies

$$v_r(x) = c\beta - \lambda e^{\frac{\lambda - c\beta}{c}x}. \tag{4.7}$$

(4.4) then follows directly. ■

Finally, we consider the deficit at ruin.

Proposition 4.3. *The distribution function of the deficit is given by*

$$\begin{aligned} P(|X(\tau)| \le z) &= \frac{\lambda(1-e^{-\beta z})}{c\beta} e^{-(\beta - \frac{\lambda}{c})x} - \frac{\lambda e^{(\lambda/c - \beta)x - \beta z}}{c\beta - \lambda}\left(e^{x\wedge z} - 1\right) \\ &\quad + \frac{c\beta - \lambda e^{\frac{\lambda - c\beta}{c}x}}{\lambda e^{\frac{\lambda - c\beta}{c}a} - \lambda e^{\frac{\lambda - c\beta}{c}b}}\left[\frac{\lambda(1-e^{-\beta z})}{c\beta}(e^{-(\beta - \frac{\lambda}{c})a} - e^{-(\beta-\frac{\lambda}{c})b})\right. \\ &\quad \left. + \frac{\lambda e^{-\beta z}}{c}\left(\frac{e^{(\lambda/c-\beta)b}}{\beta - \lambda/c}\left(e^{b\wedge z} - 1\right) - \frac{e^{(\lambda/c - \beta)a}}{\beta - \lambda/c}\left(e^{a\wedge z} - 1\right)\right)\right]. \end{aligned} \tag{4.8}$$

Proof. When $\delta = 0$ and $\omega(x_1, x_2) = \mathrm{I}_{\{x_2 \le z\}}$, the Gerber–Shiu function $m_d(x)$ as a function of z gives the distribution function of the deficit at ruin. Note that

$$\begin{aligned} \widetilde{u}_d(s) &= \frac{\lambda[s - (s+\beta)e^{-\beta z} + \beta e^{-(s+\beta)z}]}{s\beta[c(\beta+s)-\lambda]} \\ &= \frac{\frac{\lambda}{c}}{s + \beta - \frac{\lambda}{c}}\left[\frac{1-e^{-\beta z}}{\beta} - e^{-\beta z}\frac{1-e^{-sz}}{s}\right]. \end{aligned} \tag{4.9}$$

Taking the inverse Laplace transform on both sides gives

$$u_d(x) = \frac{\lambda(1-e^{-\beta z})}{c\beta}e^{-(\beta-\frac{\lambda}{c})x} - \frac{\lambda e^{-\beta z}}{c}\int_0^x e^{-(\beta - \frac{\lambda}{c})(x-u)}\mathrm{I}_{\{u \le z\}}\mathrm{d}u. \tag{4.10}$$

Note that $v_d = v_r$ is given in (4.7). The identity (4.8) then follows from (3.10). ■

As a consequence of the proposition, we can calculate the average deficit.

Corollary 4.1. *The first two moments of* $|X(\tau)|$ *are given by*

$$E_x|X(\tau)| - \frac{\lambda}{c\beta(\beta+\rho)}e^{-(\beta - \frac{\lambda\beta}{c(\beta+\rho)})x} = \frac{\frac{\lambda}{c\beta(\beta+\rho)}e^{-(\beta-\frac{\lambda\beta}{c(\beta+\rho)})a} - \frac{\lambda}{c\beta(\beta+\rho)}e^{-(\beta - \frac{\lambda\beta}{c(\beta+\rho)})b}}{\lambda e^{\frac{\lambda-c\beta}{c}a} - \lambda e^{\frac{\lambda - c\beta}{c}b}}(c\beta - \lambda e^{\frac{\lambda-c\beta}{c}x}). \tag{4.11}$$

5. The number of dividend payments before the time of ruin

Let N denote the number of dividend payments before the time of ruin. The goal of this section to find the probability distribution of N. To this end, we define the stopping time T' as the first time when the surplus reaches the level b.

Let

$$\phi(x) = \mathrm{P}_x(T' < \infty) \quad \text{and} \quad \overline{\phi}(x) = 1 - \phi(x).$$

First, we give an explicit expression for $\phi(x)$.

Proposition 5.1.

$$\phi(x) = \left(\exp\left(\frac{\lambda+\beta c}{c}b\right) - 1\right)^{-1}\left(\exp\left(\frac{\lambda + \beta c}{c}x\right) - 1\right). \tag{5.1}$$

Proof. We express the probability through conditioning on the time and amount of the first claim.

$$\begin{aligned}\phi(x) &= P_x(x + c\tau_1 \geq b) + P_x(x + c\tau_1 < b,\ T' < \infty) \\ &= \exp\left(-\frac{\lambda}{c}(b-x)\right) + \int_0^{(b-x)/c}\int_0^{x+ct}\phi(x+ct-s)\beta e^{-\beta s}ds\lambda e^{-\lambda t}dt.\end{aligned}$$

By a change of variable, we get

$$\phi(x) = \exp\left(-\frac{\lambda}{c}(b-x)\right) + \int_0^x \phi(u)I(x,u)du + \int_x^b \phi(u)J(x,u)du, \tag{5.2}$$

where

$$I(x,u) = \frac{\beta\lambda e^{-\beta(x-u)}}{\lambda+\beta c}\left(1 - \exp\left(-\frac{\lambda+\beta c}{c}(b-x)\right)\right)$$

and

$$J(x,u) = \frac{\beta\lambda e^{-\beta(x-u)}}{\lambda+\beta c}\left(\exp\left(-\frac{\lambda+\beta c}{c}(u-x)\right) - \exp\left(-\frac{\lambda+\beta c}{c}(u-x)\right)\right).$$

Taking derivative on both sides of the above equation, we get an expression for $\phi'(x)$. Substituting (5.2) into this expression, we obtain

$$\phi'(x) = \frac{\lambda}{c}\phi(x) - \frac{\beta\lambda e^{-\beta x}}{c}\int_0^x \phi(u)e^{\beta u}du. \tag{5.3}$$

Again taking derivative on both sides of the above equation, we arrive at

$$\phi''(x) = \frac{\lambda}{c}\phi'(x) + \beta\frac{\beta\lambda e^{-\beta x}}{c}\int_0^x \phi(u)e^{\beta u}du - \frac{\beta\lambda}{c}\phi(x). \tag{5.4}$$

Substituting (5.3) into above equation yields $\phi''(x) = \frac{\lambda - c\beta}{c}\phi'(x)$. Thus,

$$\phi(x) = D_1\exp\left(\frac{\lambda+\beta c}{c}x\right) + D_2. \tag{5.5}$$

The boundary conditions $\phi(b) = 1$ and $\phi(0) = 1$ then imply (5.1). ■

Theorem 5.1. *Suppose that $X(0) = x$. Then*

$$\mathrm{P}(N = n) = \begin{cases}\overline{\phi}(x), & n = 0,\\ \phi(x)\overline{\phi}(a)[\phi(a)]^{n-1}, & n = 1, 2, \ldots.\end{cases} \tag{5.6}$$

As a consequence, we have

$$\mathbb{E}(N) = \frac{\phi(x)}{\overline{\phi}(a)} \quad \textit{and} \quad Var(N) = \frac{\phi(x)(\overline{\phi}(x) + \phi(a))}{\overline{\phi}(a)^2}.$$

Proof. Suppose that $n = 0$. The event $N = 0$ means that no dividend payment happens and ruin occurs in the first period. Hence, $\mathrm{P}^x(N = 0) = \mathrm{P}^x(T' = \infty) = \overline{\phi}(x)$.

For $n \geq 1$, the event $N = n$ means that ruin occurs in the $(n + 1)$th period. This probability is then clearly given by $\phi(x)\phi(a)^{n-1}\overline{\phi}(a)$. ■

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